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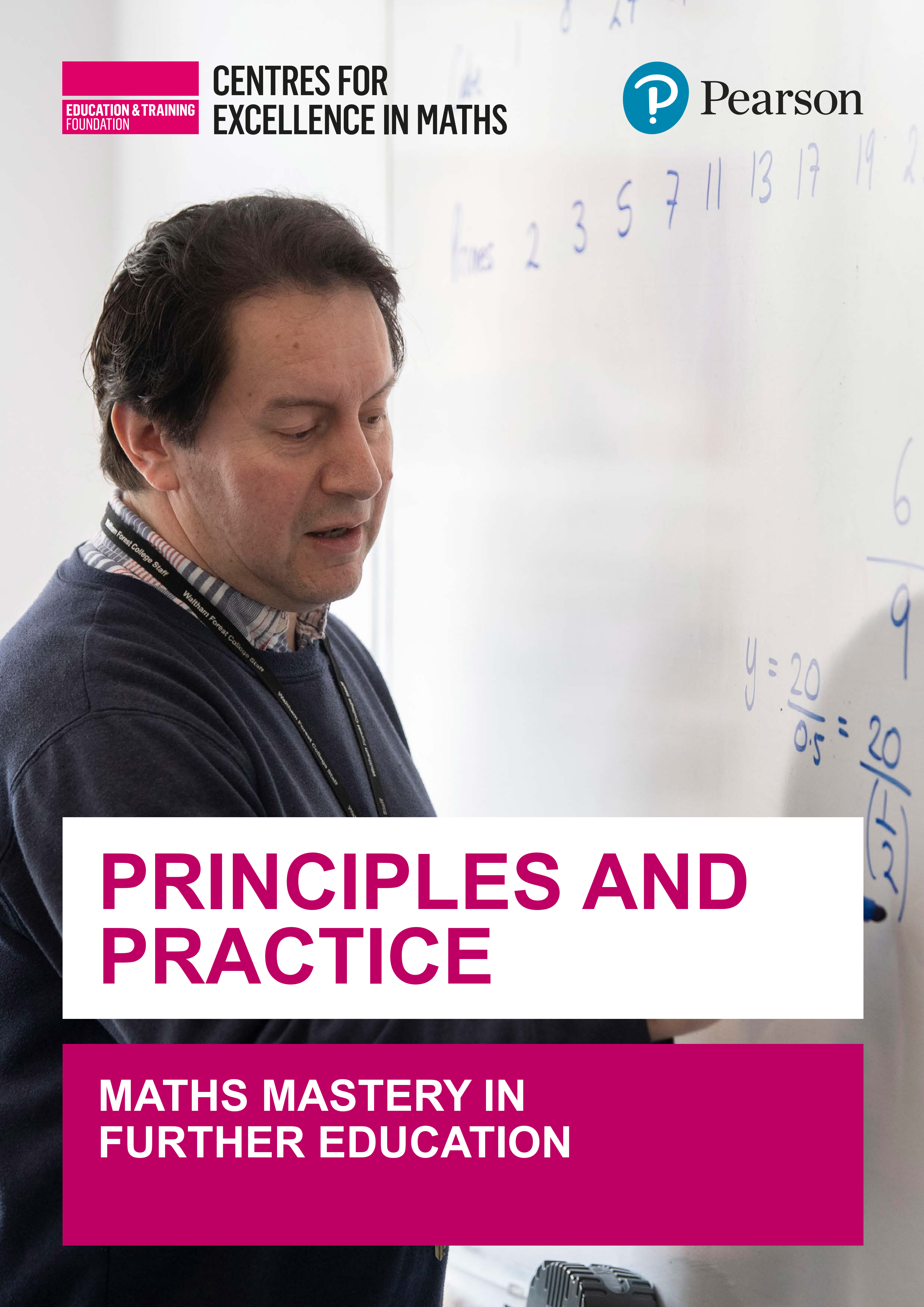


# PRINCIPLES AND PRACTICE

## MATHS MASTERY IN FURTHER EDUCATION

# Contents



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# Introduction

## Programme overview

The Centres for Excellence in Maths (CfEM) programme, funded by the Department for Education (DfE), is designed to deliver a step change in maths teaching up to Level 2 in post-16 settings. A range of expert delivery partners and Centres for Excellence across the country are working together to design and develop evidence-based teaching approaches in four themes:

- an adapted mastery approach for the post-16 sector
- contextualisation – relating maths to real-world situations
- motivating and engaging learners
- using technology and data for maths teaching.

### Centres for Excellence and delivery partners

The Centres for Excellence are 21 providers in the Further Education (FE) sector selected to drive innovation and improvement in both their institution and beyond. They are each establishing a network of ten or more partners to share practice across the FE sector. **You can see the 21 centres on the ETF website**.

The Education and Training Foundation (ETF), the national workforce development body for the Further Education and Training sector, is managing and leading the programme, working in partnership with a range of expert partners:

- **Association of Colleges (AoC)**
- **Behavioural Insights Team (BIT)**
- **Eedi**
- **Pearson**
- **PET-Xi**
- **touchconsulting**
- **University of Nottingham**
- **White Rose Maths**

### Trials and action research

An important aspect of the programme is to increase the evidence base of what works in maths teaching in the sector. The Centres for Excellence and the University of Nottingham are working together to carry out a range of trials and action research projects looking at aspects of the four themes. The focus of inquiry will change through the course of the programme.

The trials are supported by classroom resources, a professional development programme and collaborative networks. In 2019–20, the national trials are focusing on GCSE resit courses, with the key mathematical concepts outlined below ensuring that they are focused on improving attainment between GCSE Maths grades 3 and 4.

## Key mathematical concepts

The CfEM programme has focused in on a small number of key mathematical concepts. These concepts aim to provide the maximum potential improvement for students. The University of Nottingham has used anonymised exam data from Pearson Edexcel to identify the concepts that might have the most impact on attainment. Data from diagnostic questions are then used to show teachers where to target their teaching on these key concepts by highlighting common knowledge gaps and misconceptions.

The key mathematical concepts are the same across the four themes:

- working with and understanding number
- multiplicative reasoning
- fractions, decimals and percentages
- basic algebra
- measure (area and volume).

## Mastery

In the mastery theme, the focus is on applying the 'five big ideas' of mastery from the National Centre for Excellence in the Teaching of Mathematics (NCETM) to post-16 maths teaching.

# Handbooks

There is a Handbook for each of the four themes. The Handbooks are evidence-based guides for teachers on current research and good practice. They can be used by any post-16 maths teacher looking for evidence-based approaches to teaching. Each Handbook outlines research for the theme, explains why it is important to maths teaching in the post-16 sector, and exemplifies how you might consider developing your teaching to reflect some of what we know from this research.

## How to use the Handbooks

### Key principles

Each Handbook is structured around key principles. These key principles have been created to reflect both the crucial points from the research and areas where there is the potential to make the biggest difference to teaching. Developed by the delivery partners and teachers from the Centres for Excellence, these key principles provide focus for each theme and allow consistency across the themes and outputs for the programme. They are not meant to restrict how you apply the themes to your teaching. Instead, they are intended to describe how well-informed approaches might apply in each theme and support you in changing your practice in line with these approaches.

### Find out more

There are lots of links between key principles and between the different themes. 'Find out more' boxes highlight these links.

### Key terms and ideas

The most important terms and ideas in sections are highlighted in separate explanatory boxes.

### Further reading

The Further reading sections at the end of the Handbooks give you the opportunity to dig deeper into the research. These references are cited throughout the Handbooks, and particularly important documents can be found through hyperlinks.

# Mastery

## What is mastery?

When has a skill been mastered? For example, when top athletes reach the pinnacle of their sport, can they no longer improve? Do they stop making errors? Likewise, when has a mathematical skill been mastered? Is it when a student has scored full marks in a topic test? Will they necessarily still be able to show an error-free performance in a week's time, a month's time, or later?

In the context of maths, mastery needs to be distinguished from proficiency. Although the National Association of Mathematics Advisers (NAMA) notes that there is no single, clear definition of 'mastery', they also observe that there is much common ground. For example, when Benjamin Bloom wrote about mastery learning in the 1960s and 1970s his work was underpinned by the notion that all students, given enough time and effort, could succeed and improve. This remains a key aspect of the notion of mastery today.

Many of the key recommendations from a recent report produced by the Education Endowment Foundation (EEF), *Improving Mathematics at Key Stages 2 and 3*[2], link closely to the NCETM's five big ideas, outlined in the next section. As with the report, the initial focus of the NCETM's mastery work has been on developing the teaching of maths firstly in primary schools and latterly in the early years of secondary schools.[3]

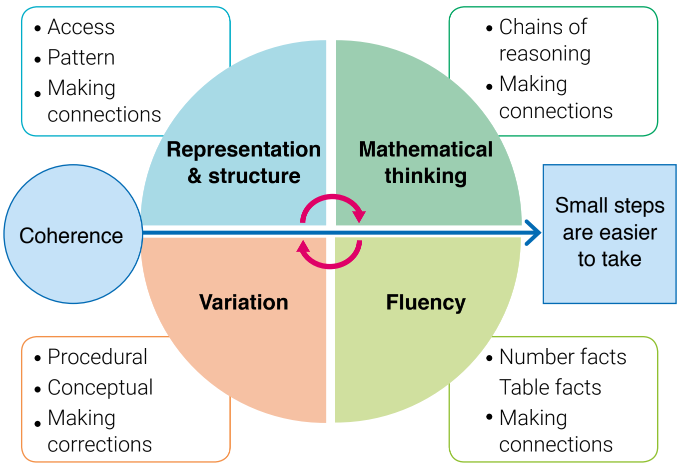
> **Mastering maths means pupils acquiring a deep, long-term, secure and adaptable understanding of the subject[1]**
>
> National Centre for Excellence in the Teaching of Mathematics (NCETM)

## Five big ideas

Since 2015, the NCETM has been advocating a mastery approach to the teaching of maths through the **Maths Hubs** programme. Heavily influenced by high-performing education systems internationally, notably Shanghai and Singapore, the NCETM model of mastery teaching centres on five connected 'big ideas'.[4]

### Key term

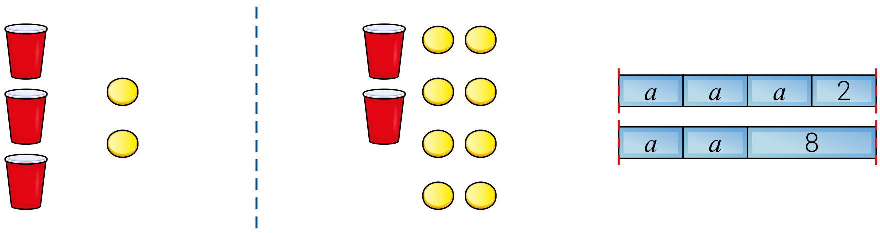
**Maths Hubs** are a countrywide network of schools and colleges driving improvements to maths education in all phases.[5]




The NCETM describes five big ideas, drawn from research evidence, that underpin mastery teaching. This diagram shows how the five big ideas complement each other.

### Representation and structure

Representation and structure means considering how different models can be used to explain one mathematical concept, and also how some models can expose the links between different areas of maths. For example, an equation can be represented concretely, pictorially or abstractly, as in the diagram below.




$3a + 2 = 2a + 8$

Concrete, pictorial and abstract representations of an equation

## Mathematical thinking

Mathematical thinking involves moving beyond the application of procedures to develop a deeper understanding of the underlying maths, and appreciating the many connections between topics and procedures. Students are expected to use correct mathematical language to describe, explain and justify their thinking. For example, they can ask questions such as 'What's the same?', 'What's different?' and 'Is this always, sometimes or never true?'.

## Fluency

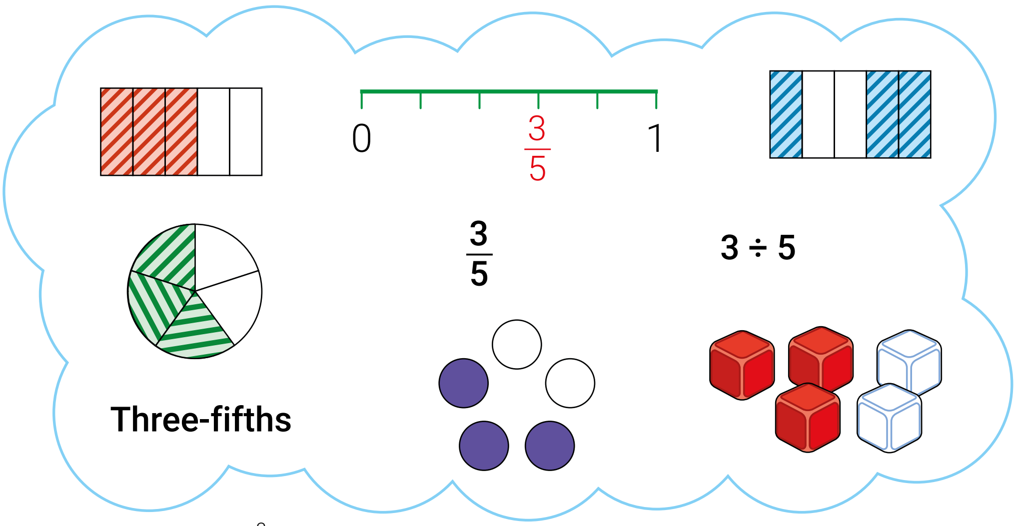
Fluency involves quick and efficient recall of facts and procedures in order to allow flexibility and free up valuable working memory. Knowing the times tables up to 12 × 12 is useful, but true fluency is then being able to work out, say, 14 × 12 using known facts; for example, (12 × 12) + (2 × 12) or (10 × 12) + (4 × 12).

## Variation

There are two types of variation: conceptual and procedural.

### Conceptual variation

Conceptual variation helps us to understand what a concept is, by representing it in many forms. For example, all the images below represent the same fraction but in different ways.




Different ways of representing $\frac{3}{5}$

### Procedural variation

Procedural variation varies one aspect of a concept at a time in order to highlight what is important and what is not. For example, consider expanding this sequence of binomials:

| | | | |
|---|---|---|---|
| $(x + 1)(x + 1)$ | $(x + 2)(x + 1)$ | $(x + 3)(x + 1)$ | $(x + 4)(x + 1)$ |
| $(x + 1)(x + 2)$ | $(x + 2)(x + 2)$ | $(x + 3)(x + 2)$ | $(x + 4)(x + 2)$ |
| $(x + 1)(x + 3)$ | $(x + 2)(x + 3)$ | $(x + 3)(x + 3)$ | $(x + 4)(x + 3)$ |
| $(x + 1)(x + 4)$ | $(x + 2)(x + 4)$ | $(x + 3)(x + 4)$ | $(x + 4)(x + 4)$ |

Students can be asked to consider relationships between the questions and answers that will expose underlying structure and later help with factorising.

## Coherence

Coherence means a well-structured journey through the curriculum using a series of small, connected steps. Considerations for a coherent curriculum include how to sequence the subject matter to make links between topics (such as fractions, decimals, percentages, ratio and proportion) and how to include a balance of all the aspects of mastery teaching.

# How can mastery be implemented in post-16 settings?

There are inevitably going to be challenges in adopting the mastery approach at post-16. Students arrive with gaps in their knowledge, they are not often fluent in calculations and may lack understanding, having relied on rote learning. The five big ideas from NCETM, however, are transferable to post-16 settings. This can be demonstrated by considering each of the four ways the term mastery is used, as described by Askew et al. (2015).[6]

## A mastery approach

This is the belief that anyone can succeed in maths. The apparent sense of failure felt by many students who have not achieved the expected standard at GCSE can have a major impact on their motivation (Higton et al., 2017).[7] Investing in the development of a 'can do' attitude is key.

## A mastery curriculum

A mastery curriculum features the development of a connected pathway. In Further Education (FE) this needs to take into account the limited time available to FE providers, the variety in timetabled hours, and the fact that much, if not all, of the curriculum has been met before.

## Mastery teaching

Pedagogic practices that teach for mastery allow students to gain proficiency and understanding. As shown by the NCETM's five big ideas, some of these strategies are already embedded in FE practice and can be developed further.

## Achieving mastery

This means that students know 'why', 'that' and 'how' and are able to use their knowledge flexibly rather than just memorise procedures. In particular, this aspect empowers students to better access the problem-solving type questions that are often omitted by students who attain lower grades.


Adopting a mastery approach for post-16 requires specific considerations to be made

# Key principles

Taking the research and recommended pedagogic ideas into account, there are five key principles that this Handbook will consider to demonstrate the implementation of mastery in an FE setting.

## Key principle 1: Mathematical structure

***1. Teaching that allows students to develop an understanding of mathematical structure***

The key point of this principle is to understand how representation can be used to unlock understanding so that students know the 'why' and not just the 'how'. Representations can both clarify the meaning of a concept and provide access to the structure of mathematical problems.

## Key principle 2: Prior learning

***2. Valuing and building on students' prior learning***

Teachers can celebrate and build on what students already know and make maximum use of the teaching time available to fill in key gaps in knowledge and understanding. There may be a number of misconceptions that need to be unpicked.

## Key principle 3: Curriculum coherence

***3. Prioritising curriculum coherence and connections***

Students need to be encouraged to see the links between mathematical concepts (for example, similarity, ratio and trigonometry) rather than seeing them as separate content that need to be individually learned. As well as these curricular links, using familiar representations across different topics develops flexibility in their use and supports problem solving.

## Key principle 4: Fluency and key ideas

***4. Developing both fluency and understanding of key ideas***

Covering key content in depth to attain fluency and understanding that can be applied in different contexts is preferable to superficial coverage of a larger amount of material. It is important to remember that fluency is not just about knowing facts and procedures, but also how and when to use them.

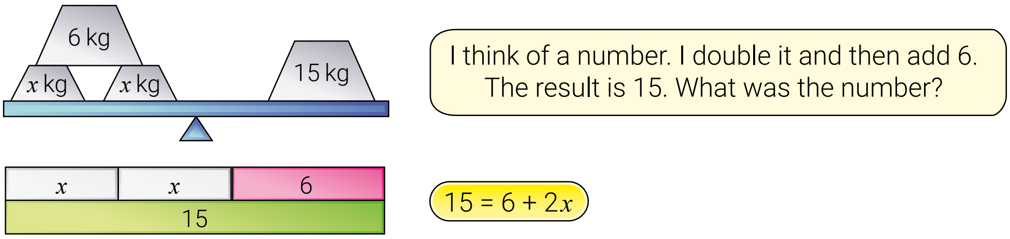
## Key principle 5: Belief in success

***5. Developing a culture in which everyone believes everyone can succeed***

The belief that effort leads to improvement can be embedded through low-threat/high-challenge tasks and activities. Success is, of course, relative; for some students it may be passing over the Grade 4 GCSE threshold, whilst for others in their first year of post-16 it may be starting their journey by moving from a Grade 1 to a Grade 2.

# Key principle 1: Mathematical structure

## Why mathematical structure?




Different ways of representing $2x + 6 = 15$

The four representations in the diagram above could be seen as very different, but a brief examination reveals that they share the same underlying structure and the solutions are identical. Paying attention to the underlying structure helps students to make connections between problems and strategies and helps them to access problems. Similarly, understanding the structure of the inverse relationships between addition and subtraction and between multiplication and division enables students to perform the calculations needed to solve the problems shown.

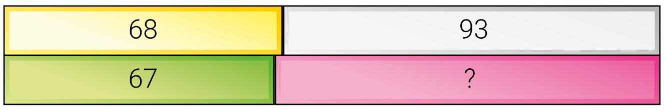
Consider these two different relationships:

$3x + 5 = 17$ $\qquad$ $3(x + 5) \equiv 3x + 15$

Understanding the different structures of the maths here is very revealing. The first is an equation which is true for only one value of $x$, which can be found, whereas the second is an identity that holds true for any value of $x$. Many students misinterpret the equals sign as an instruction to calculate – to find the answer – rather than understanding that the two expressions on either side are equal. For example, consider this problem:

$68 + 93 = 67 +$ ______

Many students would calculate the total of the left-hand side and then subtract 67 from their result. To make the expressions 'equal to' each other, the first addend has been reduced by 1, so the second needs to be increased by 1. This can be shown on the bar model below:



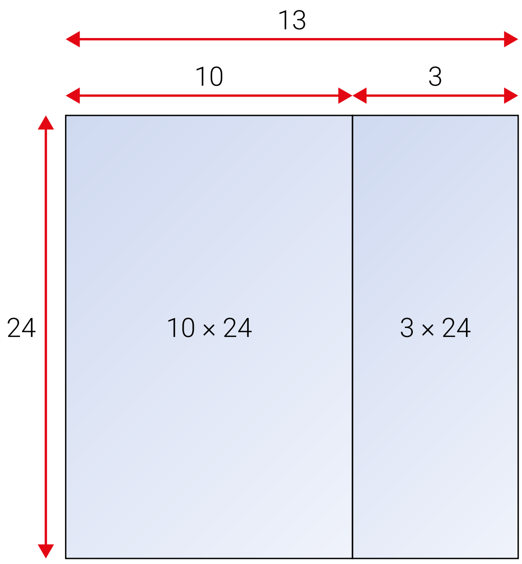

Bar model

Similarly, an area model can be used to illustrate that $13 \times 24 = (10 \times 24) + (3 \times 24)$

This can lead on to illustrate the expansion of brackets:

$a(b + c) \equiv ab + ac$

Finding appropriate representations to show the maths is key to understanding and solving problems. A key element of mastery teaching is the 'concrete, pictorial, abstract' (CPA) approach. This is where students develop their understanding of abstract mathematical concepts and structures, starting with concrete manipulatives (physical objects such as cubes and counters), moving on to pictorial representations (such as the bar model) and then linking these to abstract mathematical symbols (numerals and algebraic notation).




Area model

# What does the research show?

Bruner (1966) defined three systems of representation: enactive (actions, using *concrete* manipulatives), iconic (*pictorial*, using pictures), and symbolic (*abstract*, using words, symbols and letters).[8] This approach has been a key aspect of teaching in Singapore since the 1980s and has seen dramatic improvement in its performance in international comparative tests, such as the Programme for International Student Assessment (PISA), up to the present day.[9]

The EEF report *Improving Mathematics at Key Stages 2 and 3* references many meta-analyses that show the efficacy of manipulatives and representations, significantly noting that 'manipulatives and representations can be used to support students of all ages'.[2] Although the evidence is stronger for manipulatives, there is also clear evidence that visual representations help support problem solving for those struggling with maths.

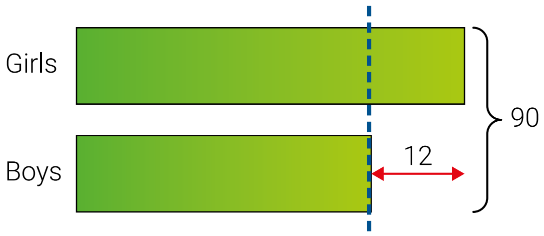
# How to put this into practice

The EEF report makes clear that 'manipulatives and representations are just tools: how they are used is essential'.[2] You need to be confident in your use and have a clear rationale for using the chosen representation to teach particular mathematical concepts.

Likewise, the use of representations must be specifically taught. For example, consider this problem:

> There are 90 students at a party.
> There are 12 more girls than boys.
> How many boys are at the party?

At first sight many teachers will see this as an algebraic problem, that could possibly be solved through simultaneous equations, but this can be easily represented as a bar model.

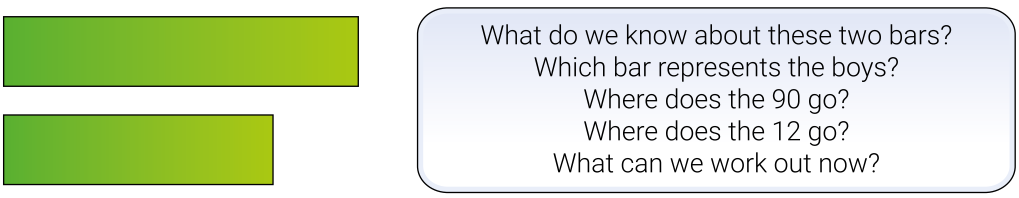



Bar model

## Find out more

Key principle 3: Curriculum coherence explores the bar model in more detail.

The bar model does not solve the problem but it shows the way forward: 90 – 12 = 78 to make two equal parts, so there are 78 ÷ 2 = 39 boys. Students are unlikely to come up with this representation independently and you will, at first, need to supply the model and use careful questioning to support them with its use.




Questions you can ask your students

As students become more experienced with the modelling they will become adept, particularly if they are used to employing similar models in other areas of the curriculum.

Many other representations are available, such as the double number line for a wide variety of proportional reasoning problems, double-sided counters for negative numbers, and so on. For most effective practice, you need to familiarise yourself with the representations and structures first and then choose when and how to use them.

# Key principle 2: Prior learning

## Why prior learning?

When students enter a post-16 setting, they have already studied maths for at least 11 years. This brings both advantages and disadvantages. Students already have some knowledge from which to build, which offers opportunities to save time by affirming and consolidating what they can already do and moving on quickly to material that still needs developing. Celebrating what students already know can also be motivating and help to address the sense of failure that may otherwise be found in a resit class. Likewise, having probably come from a wide variety of schools, students may have an assortment of different methods and approaches they can share with each other as they move forward in their study of maths.

Students may also have some misconceptions – partial or incorrect understandings of a topic or technique that lead to errors. To secure progress, these should not be ignored, but rather you should directly address them.


It is important to remember that your students are likely to be at different starting points with their maths

## What does the research show?

Higton et al. (2017) stress the need to examine prior learning to identify gaps in knowledge and develop appropriate learning plans.[7] They also note that 'peer-to-peer activities allow students with deficit in a given concept to learn from another student and allow the more advanced student to embed their own knowledge'.

Smith et al. (1994)[10] explore how misconceptions arise and the need to build on current knowledge to develop deeper understandings, unpicking where errors occur and building from there. Likewise, Ryan and Williams (2007)[11] state that correcting errors alone is unlikely to be an effective response, and that teachers need to tackle misconceptions at a deeper level.

## How to put this into practice

The extent of prior learning can be checked in several ways without the need to use large amounts of curriculum time carrying out long, formal assessments.

- Assessment for learning during lessons is key. A lesson can start with exploring what is known about a topic and careful questioning throughout can establish the class and individual needs. Multiple methods may emerge, which will give you deeper knowledge of prior understanding and encourage comparison of methods that develop students' mathematical thinking.
- Question-level analysis of students' performance in previous GCSE sittings can reveal key areas for improvement. Given the length of time since this performance snapshot was taken, care is needed in assuming all prior knowledge is secure. This can take you some time.
- Diagnostic questions are a useful tool to check understanding and can be used flexibly throughout a class. A well-structured multiple-choice question will have the correct answer and a series of wrong answers that each reveal a specific misconception. For example:

  Work out $-3 + (-2)$:

  **A** $-5$ (correct answer)

  **B** $-1$ (student has done $-3 + 2$)

  **C** $5$ (student has assumed the two negatives 'cancel each other')

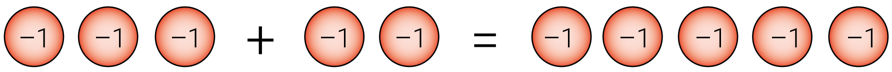
  **D** $1$ (student has done $3 + -2$)

When identified, misconceptions need to be addressed. Students may have misconceptions from their pre-16 learning, but misconceptions can also stem from learning new material. Misconceptions often arise from incorrect generalisations, for example, the notion that 'two negatives make a positive'. Sometimes this can be the case, as when subtracting a negative number from a quantity, such as 3 – (–2) = 3 + 2 = 5. Students often misapply this to any situation where two negative numbers are seen and may conclude that –3 + (–2) also gives 5.

## Find out more

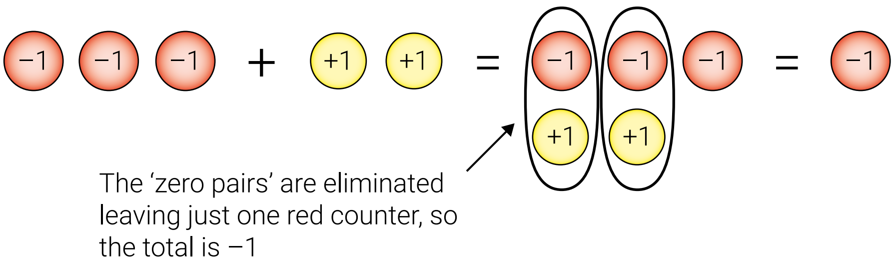
As discussed in Key principle 1: Mathematical structure, using manipulatives can support understanding and address misconceptions.

In this case, double-sided counters are used as manipulatives where one side represents –1 and the other +1 to help students visualise and understand relationships. We can represent –3 + (–2) like this:

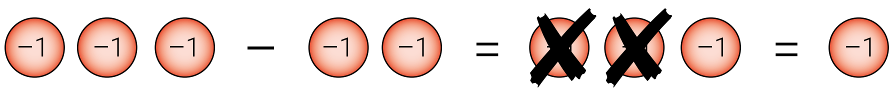



This clearly shows that –3 + (–2) = –5.

Working with 'zero pairs' based on the fact that 1 + (–1) = (–1) + 1 = 0 we can also show that –3 + 2 = –1:




To illustrate subtraction we can physically 'take away' counters to show –3 – (–2) = –1




We can establish that –3 – (–2) is equivalent to –3 + 2. Through exploration, the misconception is challenged and students have a deeper understanding of the mathematical structure, as well as something they can mentally refer to when faced with negative numbers in the future. This will be more powerful than just re-teaching a set of rules, which may be forgotten and where the misconception, having not been dealt with, may re-emerge at a later date.

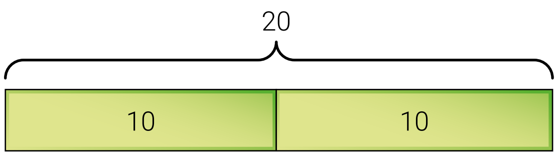
# Key principle 3: Curriculum coherence

## Why coherence?

A key issue facing post-16 providers is the limited time available to cover the content of GCSE maths. As discussed in Key principle 2, building from what students already know is important, but choices still need to be made regarding what to cover and in what depth. As well as identifying key areas, time could be saved by making the links across maths explicit to students. This allows them to use the same skills in a variety of contexts rather than seeing maths as a vast array of disparate content that needs to be learned. A mastery curriculum should be a clear pathway highlighting connections alongside coherent steps that develop understanding, each building from the last.

To support students further, using the representations discussed in Key principle 1 consistently can help students to see connections and make links for themselves.

For example, consider this very simple bar model:




This shows: 10 + 10 = 20 (addition)

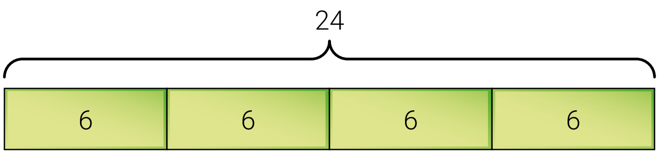
20 – 10 = 10 (subtraction)

2 × 10 = 20 (multiplication)

20 ÷ 2 = 10 (division by sharing 20 into two parts)

20 ÷ 10 = 2 (division by grouping, 'How many 10s are there in 20?')

Extending just slightly, consider this bar model:




As well as the links between addition and subtraction, repeated addition, and multiplication and division, the model can also be used to consider fractions ($\frac{1}{4}$ of 24, $\frac{2}{4}$ of 24, $\frac{3}{4}$ of 24, $\frac{4}{4}$ of 24). This can easily be extended to link other key areas of the curriculum such as ratios and percentages. Using a consistent model helps to establish links and make the journey through the curriculum more coherent for students.

## What does the research show?

Oates (2011)[12] argues curriculum coherence is more than just the order of content, but also needs to link to assessment, pedagogy and teaching materials. It is important that teaching emphasises the connections between mathematical facts, procedures and concepts.[2]

The NCETM argues that to have an effective mastery curriculum, small connected steps must be carefully sequenced, allowing students to move on to the next stage. Rosenshine (2012) states that teaching in small steps to begin with and guiding students' practice helps to resolve the limitations of working memory.[13]

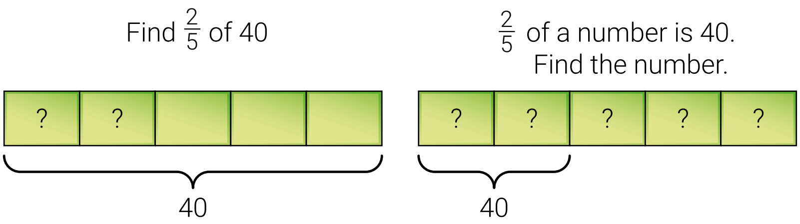
# How to put this into practice

The Introduction in this Handbook outlined the key mathematical concepts that are being focused on and have been identified as essentials. A curriculum can be built around these, carefully considering both how they link and how they can be revisited as the year goes on to support retention. It is also worth considering which topics may need to be left out in order to allow sufficient time to develop understanding of what is covered. This is a necessary adaptation to the mastery approach for use in the FE sector to address the lack of time available.
The topics covered will vary from class to class depending on students' current level of attainment and realistic target grades but, for example, it may be more appropriate to spend longer on securing students' ability to solve linear equations (and to practise this within other areas, such as probability or missing angles problems) rather than cover this superficially to allow for brief coverage of complex pairs of simultaneous equations.

It is vital to ensure that the basics are secure before working on a topic that depends on them. When planning lessons, it will be important to remember that for almost all topics, students will not be starting from scratch and so identifying the correct starting points will be vital.

## Bar models

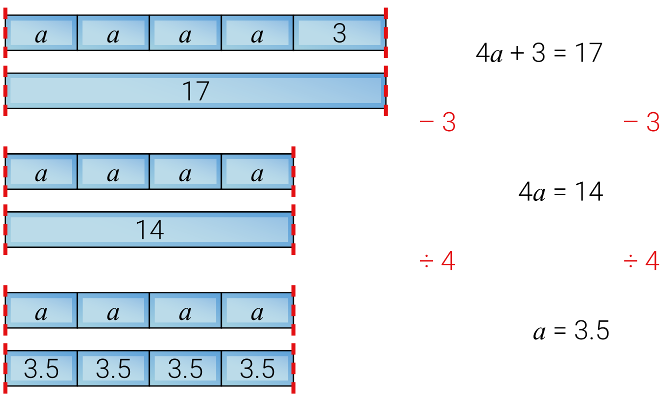
A coherent approach to the use of models can also be built into the curriculum. As discussed earlier, the bar model can be used to illustrate fractions, providing support in accessing problems which could easily be confusing. This is shown below.




The same model could be used to consider sharing in ratio, increasing or decreasing by a percentage or finding the original amount.

More than one bar can be used to compare quantities, for example, as an alternative method for sharing in a ratio (particularly useful if one part is given or if we are looking for the difference), or for modelling solving equations alongside the abstract method.

The bar models make sense of this method and support students to be able to apply the maths without the model in due course.



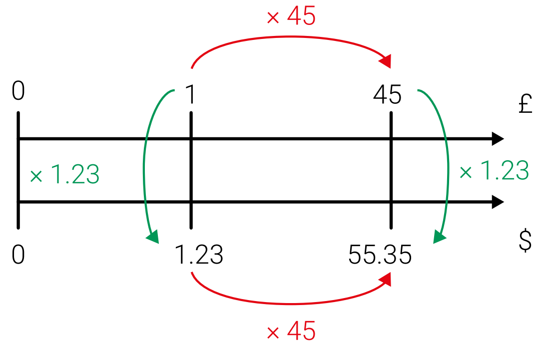

### Find out more

An example of using more than one bar to solve a word problem is illustrated in Key principle 1: Mathematical structure.

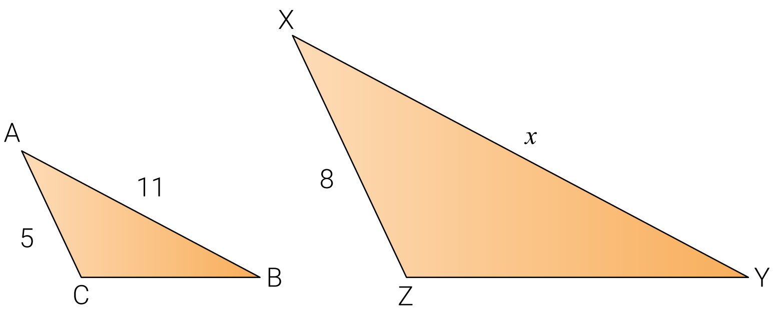
## Double number lines

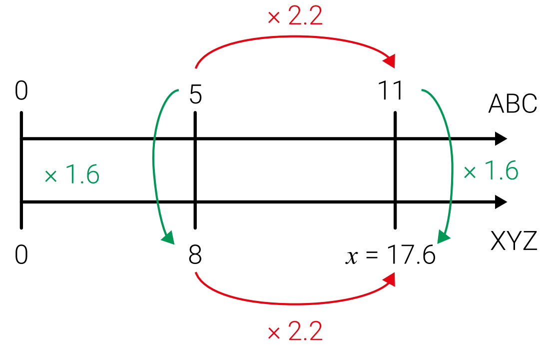
Of course, the bar model is not the only model that can be used, and nor is it suitable for every maths problem. Another key representation is the double number line. This is particularly useful for multiplicative reasoning, another key curriculum area with many links. The double number lines below illustrate their efficacy in exchange rates and similarity.

**1.** £1 = $1.23. Convert £45 to dollars.




**2.** The two triangles shown are similar.
Find the value of $x$.




Possibly more important than the choice of model is your confidence in using and demonstrating their use. Coherence will be achieved when the students are comfortable in the application of the models in a wide variety of situations.

# Key principle 4: Fluency and key ideas

## Why fluency?

One of the three key aims of the national curriculum for maths is that students 'become fluent in the fundamentals of maths, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately'.[14] Thus, fluency is much more than simple recall of facts, it is also depth of understanding and application. There are close links to Key principle 1: Mathematical structure, as understanding a variety of representations reinforces underlying mathematical structure and the ability to apply skills in unfamiliar situations and to solve problems.

Becoming fluent in basic skills is important as it frees up working memory. If, when we are solving a problem, we are weighed down by the performance of the arithmetic involved, this will take time and attention away from our journey to the solution. Similarly, the more fluency we develop with recognising concepts and ideas, the sooner we can proceed to the solution. Consider this problem:

> The table below shows the probabilities that a die lands on the numbers 1, 2, 3 and 4.
>
> | Score on die | 1 | 2 | 3 | 4 | 5 | 6 |
> |---|---|---|---|---|---|---|
> | Probability | 0.1 | 0.25 | 0.2 | 0.05 | $x$ | $2x$ |
>
> Work out the probability that the die lands on 6.

To solve the problem, you need to:

- recall that the sum of the probabilities of all outcomes of an event is 1
- know how to simplify an expression containing numbers and algebraic terms
- be able to add decimals accurately
- form and solve an equation in $x$
- multiply $x$ by 2 to obtain the required answer.

There is a lot of maths involved in this comparatively short and simple problem, and the greater the fluency with which each step can be performed, the more likely a student is to find an accurate solution promptly.

## What does the research show?

Russell (2000)[15] describes three aspects of fluency:

**1.** efficiency (easily carried out so that the logic of the strategy is not lost)

**2.** accuracy (including knowledge of number facts and relationships)

**3.** flexibility (choosing an appropriate strategy for the current problem).

Lynne McClure, writing for NRICH in 2014[16], cites Hiebert (1999)[17] stating that it is difficult for students to see the meaning in maths when they are just memorising mathematical rules. She suggests the use of manipulatives (concrete objects), talking about their work and consolidating in meaningful contexts as ways of supporting fluency.

Cognitive load theory states that instruction should be designed to develop knowledge in long-term memory while reducing unnecessary demands on working memory (Paas et al., 2003).[18] This means that fluent recall can free up capacity for students when solving problems.

# How to put this into practice

## Mathematical discussions

Mathematical talk is a strong vehicle to develop fluency and encourage flexibility in the classroom. Students could be challenged to work in pairs to consider how many ways they can find to mentally work out a calculation such as 35% of 80. Typical responses might include:

10% of 80 = 80 ÷ 10 = 8
30% of 80 = 3 × 8 = 24
5% of 80 = 8 ÷ 2 = 4
So, 35% of 80 = 24 + 4 = 28

10% of 80 = 80 ÷ 10 = 8
25% of 80 = 80 ÷ 4 = 20
So, 35% of 80 = 8 + 20 = 28

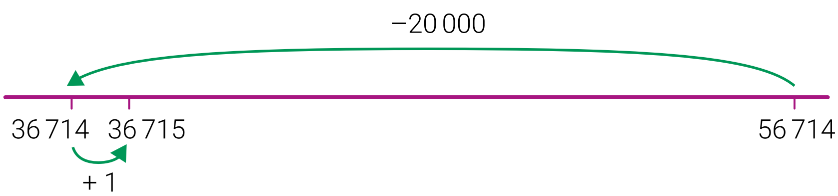
10% of 80 = 80 ÷ 10 = 8
20% of 80 = 2 × 8 = 16
5% of 80 = 8 ÷ 2 = 4
So, 35% of 80 = 8 + 16 + 4 = 28

The most common is the 'build up from 10%' method on the left. For many students this may be their only strategy, but there are many other possibilities, and students can discuss their merits in terms of efficiency.

If students are then asked to work out 95% of 160, a discussion about the most efficient approach could follow. For example, although finding 10%, multiplying by 9, then finding 5% and adding the two results together works, it is far less efficient than finding 5% and subtracting this from 160. Students could then be presented with similar questions (work out 45% of…, 75% of…, 90% of…,11% of…) where the focus of learning is not just the fluency in calculation but the choice of method.

## Fluency with calculators

It would be useful in an exercise like this to include a comparatively 'rogue' question, such as 87% of £3520 to emphasise that in some cases using a calculator is much more appropriate. Students need to be taught to use fluency when using a calculator and in making the decision of when to use mental, written or calculator methods. Often students misinterpret 'you must show your workings' in calculator-allowed exams and show steps in calculation rather than their method, both wasting time and potentially making errors. Similarly, they could attempt a calculation such as 56 714 – 19 999 using a cumbersome error-prone algorithm rather than subtracting 20 000 then adding 1.



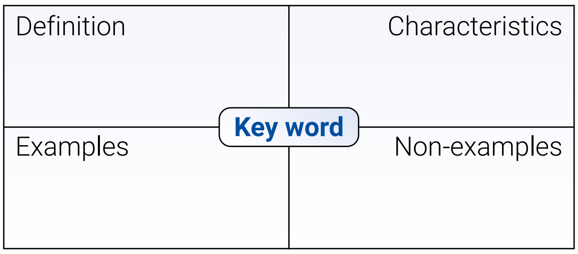

## Find out more

This example links to Key principle 1: Mathematical structure, as a number line is an effective model to illustrate this simplification of the problem.

## Using facts

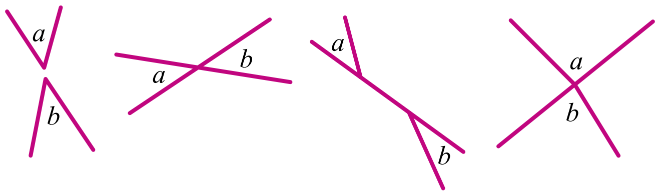
This links closely to another aspect of fluency – deriving other facts from known facts. Students could be challenged to find other facts given starting points such as $32 \times 76 = 2432$ or $4x + 3 = 21$. Activities such as these give a good insight in to current level of understanding of mathematical structure. In addition, to maintain fluency, students need to undertake regular retrieval practice to keep skills sharp.

Fluent understanding of concepts can be supported by the use of knowledge organisers such as the Frayer model.




The Frayer model

Students' understanding is established as they consider not only examples of what a concept is, but also examples of what it is not. These non-examples can then be used to focus on the characteristics of the concept and move towards a formal definition. This initially challenging task can be scaffolded by providing a series of examples and non-examples for the students to categorise, for example, for the concept 'vertically opposite angles':




Although they require investment of time, activities like these help students in their journey towards mastery; fluency and understanding of key content is preferable to superficial coverage.

### Find out more

This example links to conceptual variation, part of one of the five big ideas from NCTEM.

# Key principle 5: Belief in success

## Why is belief in success important?

There is a widely held view that maths is a subject that you are either ‘good at’ or not, and that nothing can be done to change this situation. Indeed, people from all walks of life openly discuss being ‘terrible at maths’ in the way they would rarely, if ever, say they were ‘terrible at reading’. This attitude feeds into the perception that maths is a discipline only for the gifted few, while the vast majority struggle to get by with only the basics. Post-16 resit students may already have entrenched views like these, coupled with the fact that they are, possibly through compulsion rather than through choice, taking a subject that they have already in some sense ‘failed’. Indeed, many may have been in low-attaining lesson sets for a number of years, reinforcing their negative connotations about the subject.

To challenge this, it is necessary to create a belief that everyone can succeed and that, through effort, improvement in maths is possible just as in any other challenge. The definition of success is also an issue; some students may be a long way from achieving the Grade 4 at GCSE that might be one interpretation of success. Again, the culture needs to recognise all students’ starting points and emphasise that the first step is improvement, and that this takes time. In their first year of post-16 study an improvement from, say, Grade 1 to Grade 2 is an achievement that should be celebrated as a step on the journey.

Given students’ prior experiences and attitudes, creating this culture is not easy. Shared language, understanding and a collaborative approach to learning can help to establish a low-threat/high-challenge culture that will increase students’ self-belief.

### Find out more

The CfEM Motivation and Engagement Handbook links to this key principle and has more on the importance of increasing students’ self-belief.

## What does the research show?

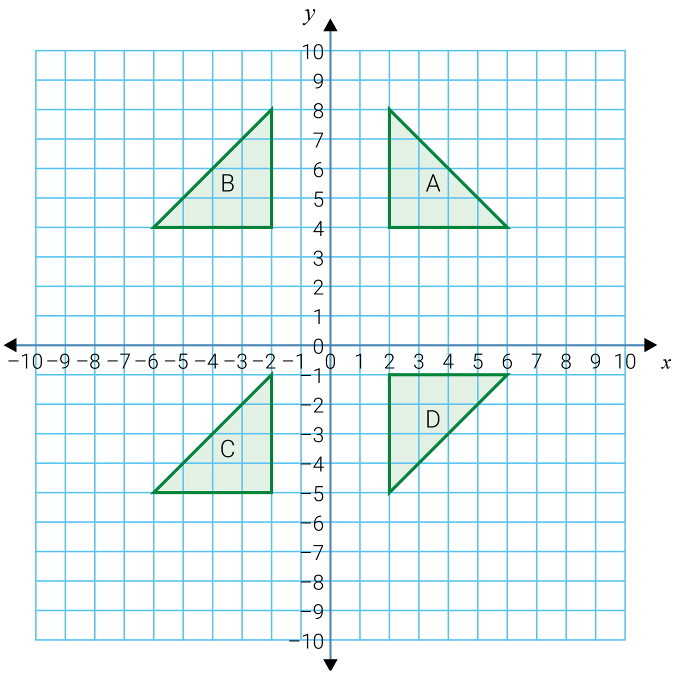
Hough et al. (2017) note that GCSE resit students experience challenges because their prior experience of learning maths has had a negative effect on their confidence and motivation.[19] Likewise, in their report *Effective practice in the delivery and teaching of English and Mathematics to 16–18 year olds*, Higton et al. (2017) state that cultivating more positive student attitudes is key, and advocate peer learning as a strategy to refine ideas and build confidence.[7]

There has been much work on developing growth mindsets, popularised by the psychologist Carol Dweck, arguing that ability is not fixed and that changing students’ and teachers’ beliefs has a positive effect.[20] Boaler (2013) also argues that notions of fixed ability are limiting, whereas growth mindset messages have a powerful impact on student attainment.[21] Boaler also argues that valuing incorrect answers is useful and helps form stronger connections in the brain.

# How to put this into practice

You need to be open and honest with students in pointing out that the study of maths involves challenges, but applying effort to challenges leads to success. Using analogies like playing a sport, learning a musical instrument or learning to drive, you can discuss how we are not born with fixed abilities in any particular skill, but by practising we can improve. You may need to consider your own mindset – do we all believe that our students can succeed?

Overcoming students' self-perceptions as 'failures' can be challenging, and many may be reluctant to engage in mathematical activity at all for fear of 'getting it wrong'. One approach to initiating engagement is to use goal-free problems. This means questions that have some of the information of a typical maths question, but with some or all of the rest removed. For example, students could be presented with this diagram:




Rather than a series of closed questions, such as 'Describe the transformation to get from shape B to shape D', the problem is presented to students as 'What can you find?'. The threat of getting a wrong answer is removed as students can name coordinates, discuss the properties of the shapes, spot different transformations, and so on. The threat is reduced further by setting the task to be done in pairs or groups, encouraging peer-to-peer collaborative learning.

This type of activity links to Key principle 2: Prior learning, as the responses give you a good understanding of what students do and do not know. They are also easily generated by taking an exam question and removing some, most or all of the text. This is a useful strategy for developing students' confidence in tackling longer GCSE questions, giving them the confidence to 'break into' questions by firstly looking at small amounts of information and gradually building up to full questions.

Many of the tasks illustrated in this Handbook could be used in a collaborative manner, with students working together to develop their confidence and self-belief, with you facilitating. As students become more confident, they will become more willing to answer questions individually, with you maintaining the low-threat culture by valuing and exploring wrong answers and comparing different methods.

# Further reading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