EC2203 QUANTITATIVE METHODS IN ECONOMICS II (1997-1998)



# Lecture 1. Sets and functions

## 1.1. Numbers

Whole numbers such as -3, -2, -1, 0, 1, 2, 3 are called integers. Ratios of integers such as -1/2, 2/5, -2/3, 5/4, and 7/3 are called fractions. Observe that any integer n can be thought of as a fraction, since n=n/1. Since both integers and fractions can be expressed as ratios, they are collectively called rational numbers. Some numbers cannot be expressed as ratios of integers eg. $\pi$ and $\sqrt{2}$. These are called irrational numbers.

Imagine a ruler of infinite length, whose midpoint is 0 :


Every 'point' on this ruler is either a rational or an irrational number. The rational and irrational numbers thus form a 'continuum' which is called the set of all real numbers. The set of real numbers is often denoted by $R^1$ or $E^1$, and because it can be displayed as a straight line (as above), it is referred to as the real line.

You should already be familiar with the basic maths of real numbers (addition, subtraction, multiplication, division, inequalities, powers, logarithms etc..) from QMI, and such knowledge is assumed in what follows. Other types of numbers (eg. complex numbers) will not be discussed in this course.

## 1.2. Sets

A set is just a collection of distinct objects (eg. numbers, cars, apples). These objects are called the elements of the set. We can define a set by listing its elements eg. $S = \{1, 2, 3, 4\}$, or by describing a property that its elements satisfy eg. $T = \{x| x^2 < 1\}$. In words, 'T is the set of all real numbers x whose square is less than 1'. Membership in a set is indicated by the symbol $\in$. Thus, for the two sets S and T defined above, $2 \in S$ and $1/2 \in T$. The number 8 is not an element of S, and we write this as $8 \notin S$. If $E^1$ denotes the set of all real numbers, then 'x is a real number' can be written as $x \in E^1$.

### 1.2.1. Set inclusion

If all the elements in a set U are also in the set V, we write $U \subset V$, and say that U is a subset of V. For example, if $U = \{2, 4, 6\}$ and $V = \{x| \text{ x is an even number}\}$, then $U \subset V$. If two sets W and X have identical elements, then $W \subset X$ and $X \subset W$, and we write W=X. An important example of a subset is that of an interval of the real line eg. the set $\{x| 3<x<9\}$ is a subset of $E^1$, so we write $\{x| 3<x<9\} \subset E^1$.

The null set or empty set (denoted by $\varnothing$) contains no element at all, and is considered to be a subset of any set that can be conceived. The universal set (denoted by $\Omega$) is the set of all objects that are relevant in a particular context of discussion, and every set in that context is assumed to be a subset of $\Omega$ (see below).

### 1.2.2. Union, intersection and complementation

The union of two sets A and B, denoted by $A \cup B$, is the set of all elements belonging to A, or to B, or to both A and B. For example, if $A = \{3, 5, 7\}$ and $B = \{2, 3, 4, 8\}$, then $A \cup B = \{2, 3, 4, 5, 7, 8\}$. The intersection of A and B, denoted by $A \cap B$, is the set of all elements belonging to both A and B. For example, using the definitions of A and B above, $A \cap B = \{3\}$.

Recall that the universal set $\Omega$ is the set of all objects that are relevant in a particular context of discussion. The complement of a set A in $\Omega$, denoted by $A^c$, is the set of all elements in $\Omega$ which are not in A. Continuing the above example, we might define $\Omega$ as the set of the first eight positive

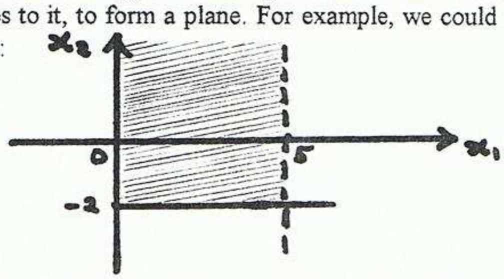
integers ie. $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Using the definitions of A and B above, we then have $A^c = \{1, 2, 4, 6, 8\}$, $B^c = \{1, 5, 6, 7\}$, and $\Omega^c = \varnothing$.
More formally, the intersection of A and B is defined as $A \cap B = \{x | x \in A \text{ and } x \in B\}$, the union of A and B is defined as $A \cup B = \{x | x \in A \text{ or } x \in B\}$, and the complement of a set A in $\Omega$ is defined as $A^c = \{x \in \Omega | x \notin A\}$.

### 1.2.3. Cartesian product

The Cartesian product of two sets, say S and T, is the set of all ordered pairs (a, b) constructed by taking the first element, a, from S and the second, b, from T, and is written $S \times T$. Formally, $S \times T$ is defined as $S \times T = \{(a, b) | a \in S, b \in T\}$. For example, suppose $S = \{3, 4\}$ and $T = \{5, 8\}$. Then $S \times T = \{(3, 5), (3, 8), (4, 5), (4, 8)\}$.
Note that ordered pairs like (8, 4) are not in $S \times T$ as the first element in (8, 4) is taken from T. (Of course, (8, 4) is in $T \times S$).

### 1.2.4. Cartesian products and Euclidean spaces

We can use the concept of the Cartesian product of sets to 'build up' Euclidean spaces of any dimension from the real line $E^1$. Whereas 'points' in $E^1$ are real numbers, 'points' in two-dimensional space, $E^2$, are two-dimensional vectors or ordered pairs of real numbers of the form $(x_1, x_2)$. For example, the vector (2, 6) is a 'point' in 2-space. Similarly, 'points' in three-dimensional space $E^3$ (the space we live in!) are three-dimensional vectors or ordered triples of real numbers of the form $(x_1, x_2, x_3)$. For example, the vector (2, 8, 3) is a 'point' in 3-space. For any $n>3$, one can imagine (although not draw!) n-dimensional Euclidean space, $E^n$, as consisting of all vectors with n elements of the form $(x_1, x_2, \ldots, x_n)$ where each $x_i$ is a real number. To see this, use the formal definition of Cartesian product to write $E^2 = E^1 \times E^1 = \{(x_1, x_2) | x_1 \in E^1, x_2 \in E^1\}$. In words, '$E^2$ is the set of all ordered pairs whose first element is a real number, and whose second element is a real number'. Geometrically, constructing $E^2$ from $E^1$ involves adding another axis onto the real line at right angles to it, to form a plane. For example, we could draw the set $S = \{(x_1, x_2) | x_1 < 5, x_2 \geq -2\}$ as follows:




Analogously, $E^3 = E^1 \times E^1 \times E^1 = \{(x_1, x_2, x_3) | x_1 \in E^1, x_2 \in E^1, x_3 \in E^1\}$, and more generally, $E^n = E^1 \times E^1 \times E^1 \times \cdots \times E^1 = \{(x_1, x_2, \ldots, x_n) | x_1 \in E^1, x_2 \in E^1, x_3 \in E^1, \ldots, x_n \in E^1\}$. In the rest of the course, we shall use the phrases 'point in $E^n$' and 'n-dimensional vector' interchangeably.

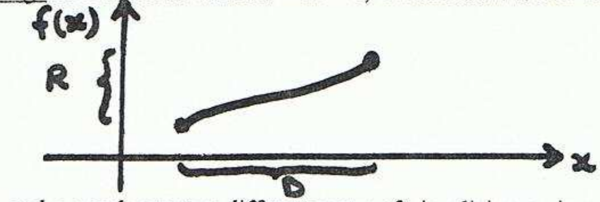
### 1.2.5. Index notation

Consider the eight mathematical quantities $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$. Thought of together, they comprise a set $S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. The subscripts are called indices, and are simply a convenient label. We can call a typical member of the set $x_i$, and we can thus write $S = \{x_i\}$, $i = 1, 2, \ldots, 8$. Double indexation can also be used. For example, suppose $S = \{a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}\}$. We can call a typical member of the set $a_{ij}$, and thus write $S = \{a_{ij}\}$, $i = 1, 2$; $j = 1, 2, 3$.

**1.3. Functions**

A function is a rule which associates with each point x in its domain, $D \subset E^n$, a single point f(x) in its range, $R \subset E^m$. We write this $f : D \rightarrow R$, and illustrate it by the bubble diagram:

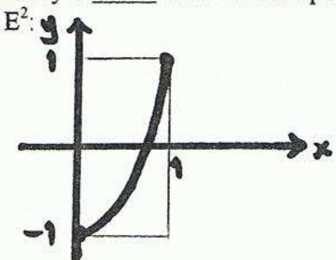

We say that f(x) is the image of x under f. When n = m = 1, we can draw a two-dimensional graph of the function eg.




(Note: a bubble diagram and a graph are two different ways of visualising a given function. Often, the bubble diagram is more useful, as the graph may be difficult (or impossible) to draw for higher dimensions). Consider, for example, the function $f(x) = 2x^2 - 1$, with domain $D = \{x | 0 \leq x \leq 1\}$ and range $R = E^1$. One way of picturing this function is the following. First, draw up a table of values, listing in the first row several values of x in the domain $D = \{x | 0 \leq x \leq 1\}$, and in the second row the corresponding images in the range $R = E^1$:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| f(x) | -1 | -0.98 | -0.92 | -0.82 | -0.68 | -0.5 | -0.28 | -0.02 | 0.28 | 0.62 | 1 |

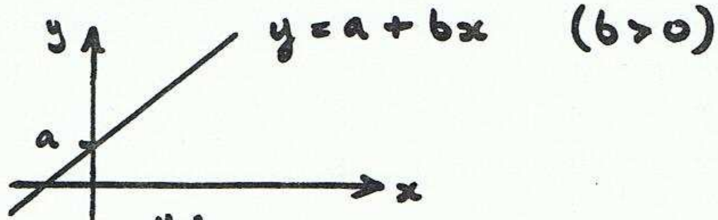
Each column in the table is essentially a vector or an ordered pair of the form (x, f(x)), which can be plotted as a point in the plane $E^2$:



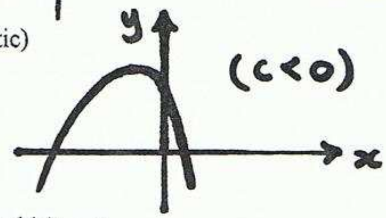

It is often convenient to think of f(x) simply as the y-coordinate corresponding to a typical point x, in which case the graph of $f : D \rightarrow R$ can be expressed as $\{(x, y) | x \in D, y = f(x)\}$. So we can label the graph simply 'y = f(x)'. We can talk interchangeably about 'the graph of f', 'the graph of f(x)', and 'the graph y = f(x)'. (Note: it is not possible to plot all the points of the graph for a function like f, which has infinitely many values of x in its domain. Fortunately, plotting a few points usually provides a good idea of what the function looks like. For any so-called 'well-behaved' function, these points can be joined up with a smooth curve - and extended if necessary - in order to complete the picture).

### 1.3.1. Examples

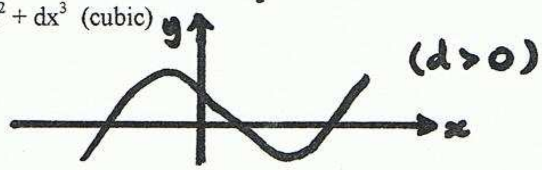
(i) $f(x) = a + bx$ (linear)



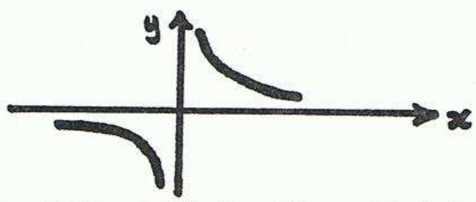

(ii) $f(x) = a + bx + cx^2$ (quadratic)




(iii) $f(x) = a + bx + cx^2 + dx^3$ (cubic)

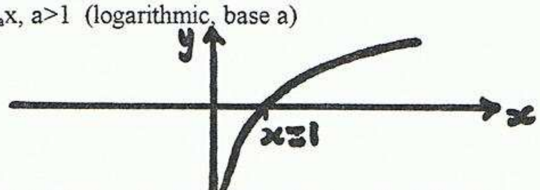



(iv) $f(x) = 1/x$ (hyperbolic)




(Note: D cannot include 0 since 1/0 is not defined, and R cannot include 0 since 1/x is never equal to zero).

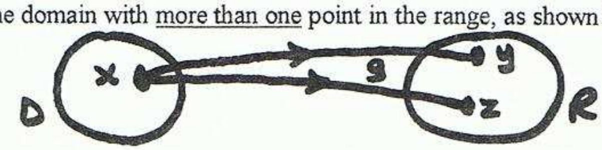
(v) $f(x) = \log_a x$, $a>1$ (logarithmic, base a)




(Note: $D = \{x | x>0\}$ since the logarithm of x is only defined for positive x).

### 1.3.2. Functions and correspondences

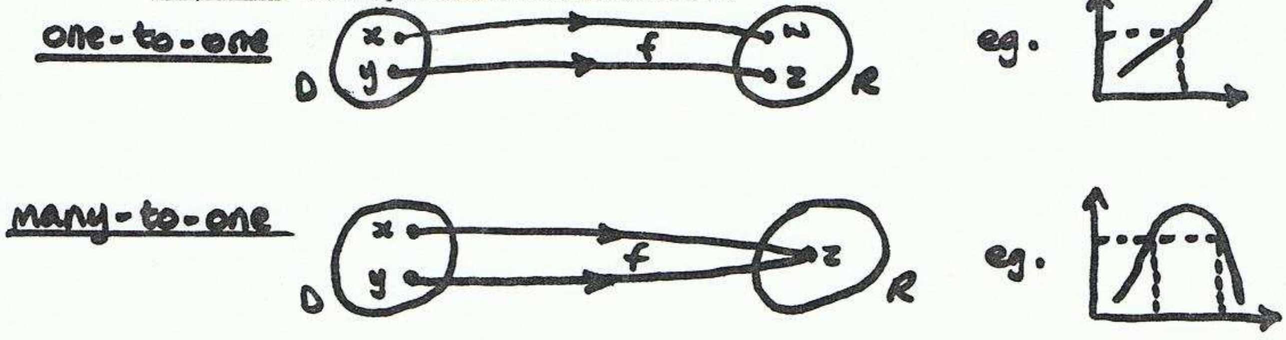
The crucial point about the definition of a function is that it associates exactly one point in the range with each point in the domain. However, it is possible to imagine rules which associate points in the domain with more than one point in the range, as shown in the diagram below:



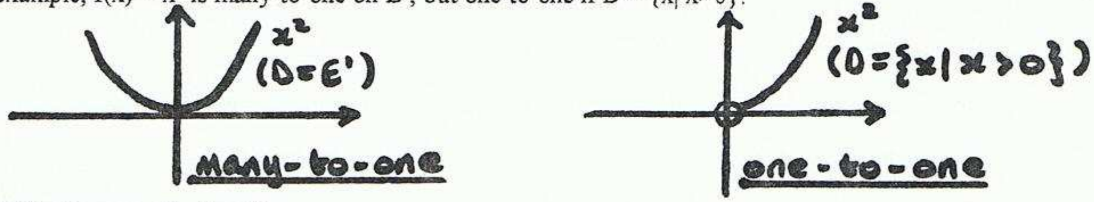

Such a rule is not a function; it is called a correspondence. For example, the rule $g(x) = \{-\sqrt{x}, \sqrt{x}\}$ which associates with each $x > 0$ both its positive and negative square root is a correspondence.

**1.3.3. One-to-one and many-to-one functions**

One can also distinguish between two types of function; those that map each and every point in the domain onto a different point in the range, and those that map two or more points in the domain onto the same point in the range. The former type of function is called one-to-one, and the latter many-to-one. The two possibilities are illustrated below:

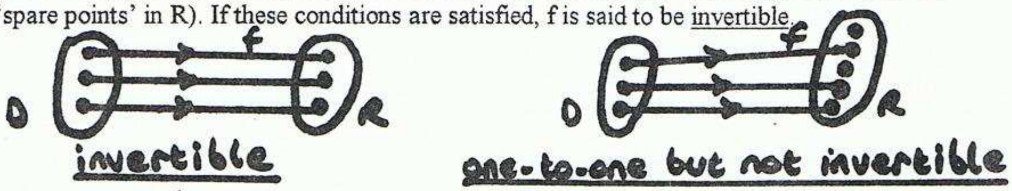



Important: whether a function is many-to-one or one-to-one depends on its domain of definition. For example, $f(x) = x^2$ is many-to-one on $E^1$, but one-to-one if $D = \{x| x>0\}$.




**1.3.4. The inverse of a function**

We now come to the important concept of the inverse of a function. A function $f : D \to R$ has an associated inverse function $f^{-1} : R \to D$ if and only if it satisfies the following two conditions: (i) f is one-to-one on D (ii) each and every point in R is linked to a point in D by the function f (ie. there are no 'spare points' in R). If these conditions are satisfied, f is said to be invertible.



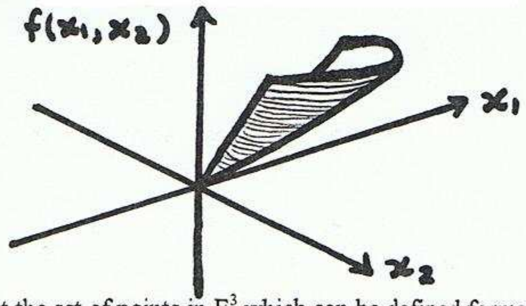

The inverse function $f^{-1} : R \to D$ is a rule which assoociates with each and every point y in the range R the point x in the domain D which the function f maps onto y. Given an invertible function $y = f(x)$, its inverse can be found simply by solving for x in terms of y to get $x = f^{-1}(y)$. For example, consider the linear function $y = f(x) = a + bx$. Solving for x in terms of y gives $x = f^{-1}(y) = (y-a)/b$.

**1.3.5. Functions of several variables and vector-valued functions**

So far, we have mainly considered functions from the real line to the real line (ie. where $D \subset E^1$ and $R \subset E^1$). The main reason for this is that such functions are easy to visualise and work with, by drawing graphs. The idea of the graph of a function generalises to higher dimensions, however. For example, suppose that f is a function of two variables (ie. from $E^2$ to $E^1$) defined by

$$f(x_1, x_2) = x_1^2 + x_2^2$$

The graph of this function is the set of points traced out in 3-dimensional space (or '3-space') as $x_1$ and $x_2$ vary ie. the graph is a surface in 3-space:




This 'surface' is in fact the set of points in $E^3$ which can be defined formally as

$$\{(x_1, x_2, x_3) \in E^3 | \ x_3 = x_1^2 + x_2^2\}$$

This formal definition of a graph in 3-space generalises to higher dimensions. The graph of a function $f : D \in E^n \rightarrow R \in E^m$ is a set in $E^{n+m}$ of the form

$$\{(\mathbf{x}, \mathbf{y}) \in E^{n+m} | \ \mathbf{y} = f(\mathbf{x}) \text{ and } \mathbf{x} \in D\}$$

Therefore, if f goes from $E^4$ to $E^5$, its graph lies in 9-dimensional space, $E^9$! It is, of course, impossible to visualise graphs beyond 3-dimensional space.

Finally, some terminology. Consider the function $f : D \in E^n \rightarrow R \in E^m$. If m=1 and n≥1, then f is a real-valued function of one or more variables eg. n=3, m=1 $f(x_1, x_2, x_3) = x_1x_2 + 4x_3^2$. If n=1 and m>1, then f is a vector-valued function of one variable (ie. points in the range are m-vectors) eg. n=1, m=3 $f(x) = (2x, 3x + 4, 7x^2)$. If m>1 and n>1, then f is a vector-valued function of many variables eg. n=2, m=2 $f(x_1, x_2) = (4x_1 + x_2, x_1x_2)$.

## 1.4. Terminology of logic

The following terminology of logic appears frequently in economics. Suppose we have six statements $S_i$ (i = 1, 2, ..., 6):

$S_1$: *The sun is shining*
$S_2$: *It is daytime*
$S_3$: *William Hague is boring*
$S_4$: *All politicians are boring*
$S_5$: *There are less than 30 days in this month*
$S_6$: *It is the month of February*

### 1.4.1. 'IF' relationships

Consider statements $S_1$ and $S_2$. Obviously, $S_2$ must be true if $S_1$ is true, but $S_2$ can also be true when $S_1$ is not true (eg. when it is cloudy). This logical relationship can be expressed as

'if $S_1$, then $S_2$'
or '$S_1$ implies $S_2$'
or $S_2 \Leftarrow S_1$
or '$S_2$ if $S_1$'
or 'a sufficient condition for $S_2$ is $S_1$'

### 1.4.2. 'ONLY IF' relationships

Consider statements $S_3$ and $S_4$. Obviously, $S_4$ cannot be true if $S_3$ is not true, because William Hague is a politician. But $S_4$ is not guaranteed even if $S_3$ *is* true, because Tony Blair might be really interesting. We can express this as

'if $S_4$, then $S_3$'
or '$S_4$ implies $S_3$'

or $S_4 \Rightarrow S_3$
or '$S_4$ only if $S_3$'
or 'a necessary condition for $S_4$ is $S_3$'

**1.4.3. 'IF AND ONLY IF' relationships**

Consider statements $S_5$ and $S_6$. Clearly, if $S_5$ is true then $S_6$ must also be true. Conversely, if $S_6$ is true, it must be the case that $S_5$ is also true. We can express this as

'$S_5$ is equivalent to $S_6$'
or '$S_5$ implies and is implied by $S_6$'
or $S_5 \Leftrightarrow S_6$
or '$S_5$ if and only if $S_6$'
or 'a necessary and sufficient condition for $S_6$ is $S_5$'

**1.5. What you must do before the next lecture (Thursday, 16th October 1997, 'Vectors')**

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter Two (pages 7-32) carefully, and make sure you are familiar with all the terms and concepts in that Chapter, and in these lecture notes. To check your understanding, try some of the problems in Chapter Two (I leave this to your discretion; the answers to many of the problems are given in an appendix at the back of the book). If you find yourself struggling with any particular areas, ask a friend to help you or come and see me in my office immediately. From next week onwards, this material will be assumed known, and will certainly crop up again in lectures, assignments and class exams. **There will be a short written test at the start of class next week (Monday 13th October or Tuesday 14th October, depending on which class group you are in) and you will be required to hand in your answers for assessment.** In the remainder of the class next week, we will cover some more essential material prior to the main lecture on Thursday (further handouts will be distributed, so make sure you attend!). Here are some worked examples of questions you might be asked in the test:

*Example 1 If the sets A and B are as specified below, find $A \cup B$ and $A \cap B$. In each case, $A \subset E^2$ and $B \subset E^2$.*

*(a). $A = \{(x_1, x_2) | x_1 \geq 1\}$  $B = \{(x_1, x_2) | x_1 \leq 1\}$*
*(b). $A = \{(x_1, x_2) | x_1 \geq 1\}$  $B = \{(x_1, x_2) | x_2 \geq 1\}$*
*(c). $A = \{(x_1, x_2) | x_2 \geq x_1^2\}$  $B = \{(x_1, x_2) | x_2 = 0\}$*

Solution: (a). $A \cup B = E^2$ ie. all 2-dimensional vectors of the form $(x_1, x_2)$; $A \cap B = \{(x_1, x_2) | x_1 = 1\}$.
(b). $A \cup B = \{(x_1, x_2) | x_1 \geq 1 \text{ or } x_2 \geq 1\}$; $A \cap B = \{(x_1, x_2) | x_1 \geq 1 \text{ and } x_2 \geq 1\}$.
(c). $A \cup B = \{(x_1, x_2) | x_2 \geq x_1^2 \text{ or } x_2 = 0\}$; $A \cap B = \{(0, 0)\}$.

*Example 2*

*(a). Suppose that f is the following real-valued function of one variable (ie. from $E^1$ to $E^1$) : $f(x) = 2x^3 - 5x^2 + 8x - 20$. Find f(5).*
*(b). Suppose that f is the following real-valued function of three variables (ie. from $E^3$ to $E^1$): $f(x_1, x_2, x_3) = x_1x_2 + 4x_3^2$. Find f(1, 2, 3).*
*(c). Suppose that f is the following two-dimensional vector-valued function of two variables (ie. from $E^2$ to $E^2$): $f(x_1, x_2) = (4x_1 + x_2, x_1x_2)$. Find f(2, 3).*

Solution: (a). f(5) = 145 (b). f(1, 2, 3) = 38 (c). f(2, 3) = (11, 6) **(End of lecture 1)**