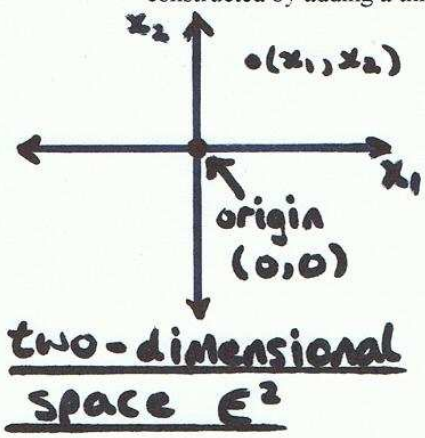
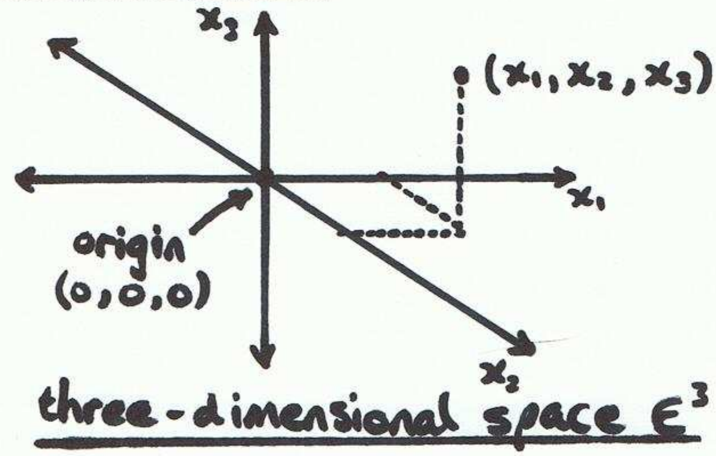
EC2203 QUANTITATIVE METHODS IN ECONOMICS II (1997-1998)

Lecture 2. Vectors

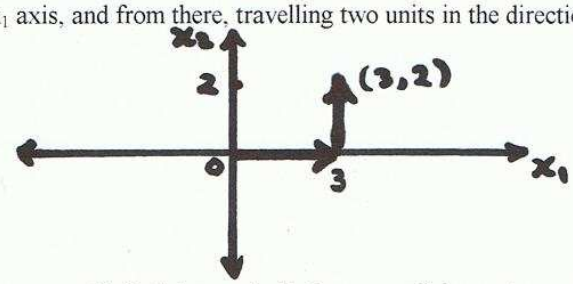
2.1. The length and direction of a vector

We said in the last lecture that a vector with n elements of the form $x = (x_1, x_2, ..., x_n)$ can be viewed as a 'point' in n-space. If n=1, the point lies somewhere on the real line E^1. If n=2, the point lies somewhere on the two-dimensional plane E^2 constructed from E^1 by adding another axis onto the real line at right angles to it. If n=3, the point lies somewhere in the three-dimensional space E^3 constructed by adding a third axis at right angles to the previous two:





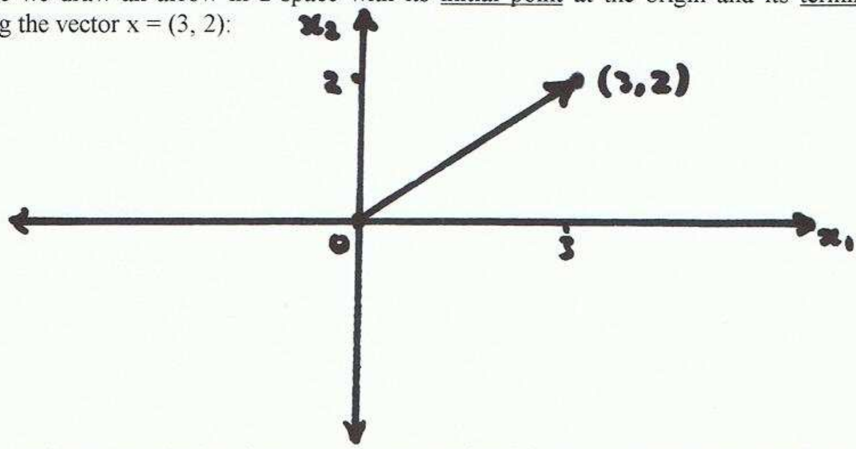
If n>3, we can still talk of the vector as a 'point' in an n-space constructed by having n 'axes' simultaneously at 'right angles' to each other, but we cannot visualise these axes or draw them (although they most definitely do exist!). In all cases, the axes intersect at a single point called the origin (see the diagrams above). We think of the elements of the vector $x = (x_1, x_2, ..., x_n)$ as the 'co-ordinates' of x in n-space, as viewed from the origin. For example, the vector $x = (3, 2)$ is a point in 2-space which can be reached from the origin firstly by travelling three units in the direction of the x_1 axis, and from there, travelling two units in the direction of the x_2 axis:



Similarly, the vector $x = (3, 2, 4)$ is a point in 3-space which can be reached from the origin firstly by travelling three units in the x_1 direction, then (from there) by travelling two units in the x_2 direction, and finally from there travelling four units in the x_3 direction. The same applies to the general n-vector $x = (x_1, x_2, ..., x_n)$; it is just that we cannot visualise travelling in n different directions (along paths which are all 'perpendicular' to each other) to reach the point x in n-space from the n-dimensional origin.

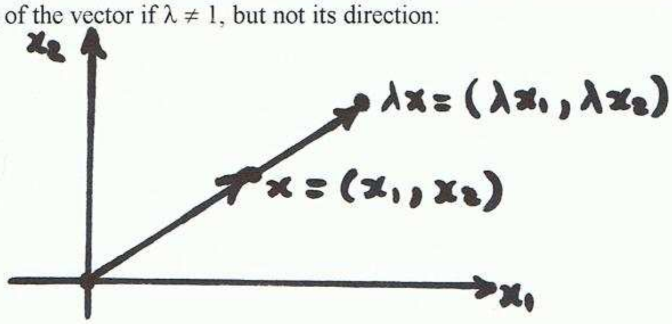
Note that the origin is itself a vector, but one whose elements are all zero (a vector with all elements equal to zero is called a null vector). For example, the origin in E^2 is the two-dimensional

null vector (0, 0). The origin in E^3 is the three-dimensional null vector (0, 0, 0). The origin in E^4 is the four-dimensional null vector (0, 0, 0, 0), etc...
Suppose we draw an arrow in 2-space with its initial point at the origin and its terminal point touching the vector x = (3, 2):

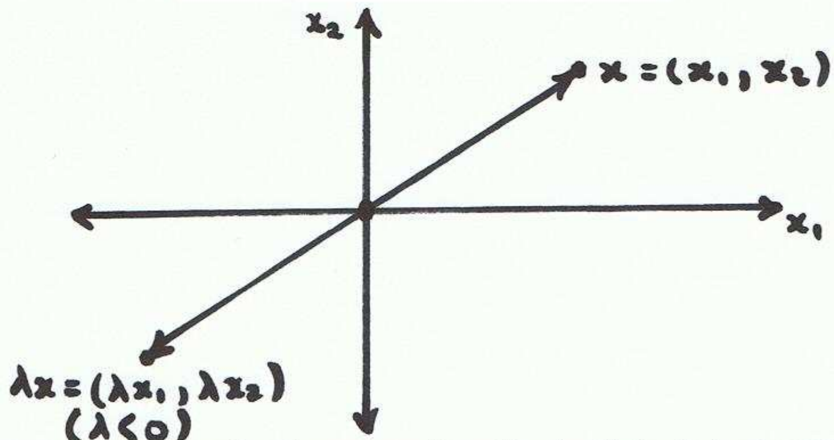


This arrow has a certain length (you can measure it with a ruler!) and a certain direction. The length and direction as viewed from the origin are together unique to the point (3, 2). We can therefore think of x as being represented graphically either by the point (3, 2), or by the corresponding arrow emanating from the origin.
In exactly the same way, we can think of the n-vector $x = (x_1, x_2, ..., x_n)$ as being represented either by an arrow or a point in n-space, but we cannot visualise either of these for n>3. Nevertheless, we still say that the vector has a length and a direction which together are unique to that vector.

2.2. Scalar multiplication
In the terminology associated with the study of vectors and matrices, real numbers (ie. 'points' in E^1) are called scalars. A vector $x = (x_1, x_2, ..., x_n)$ in E^n multiplied by a scalar λ gives the vector $\lambda x = \lambda(x_1, x_2, ..., x_n) = (\lambda x_1, \lambda x_2, ..., \lambda x_n)$. In words, scalar multiplication of a vector involves multiplying every element of the vector by the same scalar. For example, if the vector is (3, 2) and $\lambda = 2$, then 2(3, 2) = (6, 4). Geometrically, multiplying a vector by a positive scalar (ie. $\lambda > 0$) changes the length of the vector if $\lambda \neq 1$, but not its direction:



Multiplying a vector by a negative scalar (ie. $\lambda < 0$) reverses its direction, as well as changing its length if $\lambda \neq -1$:



Multiplying a vector by zero gives the corresponding null vector. Notice that scalar multiplication always results in a vector that lies on the same 'line' through the origin. In fact, λ is called a scalar because it 'scales' up or down the vector along the same 'line'.

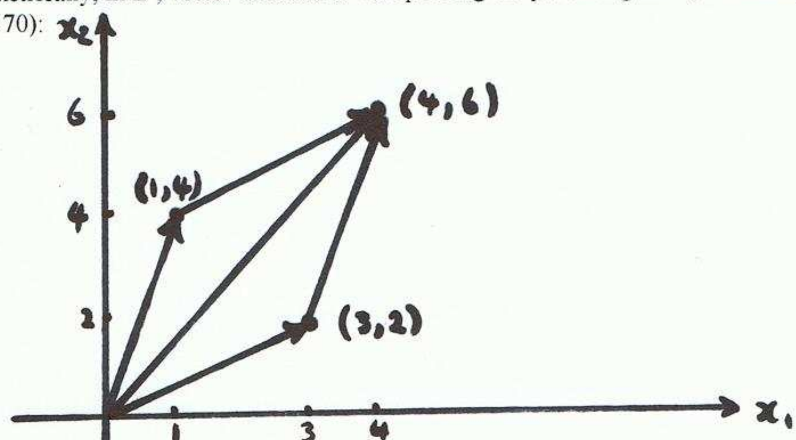
2.3. Vector addition

The sum of two vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ is defined to be

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

In other words, x+y is the vector produced by summing together corresponding elements in the vectors x and y. For example, (3, 2) + (1, 4) = (4, 6). Clearly, the sum x + y cannot exist unless the vectors x and y are of the same dimension.

Geometrically, in E^2, vector addition is 'completeing the parallelogram' (see Chiang, Chapter Four, page 70):



2.4. Linear combinations

The following is one of the fundamental concepts from this lecture that you must make sure you understand fully before the start of next week.

A vector $z = (z_1, z_2, ..., z_n)$ in E^n is called a linear combination of the vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ if it can be expressed in the form

$$z = \lambda x + \mu y = (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, ..., \lambda x_n + \mu y_n).$$

For example, the vector (10, 20) in E^2 is a linear combination of the vectors (3, 2) and (1, 4) because

$$(10, 20) = 2(3, 2) + 4(1, 4) = (6 + 4, 4 + 16)$$

As another example, the vector $z = (9, 2, 7)$ in E^3 is a linear combination of the vectors $x = (1, 2, -1)$ and $y = (6, 4, 2)$ because $z = -3x + 2y$.
One can, of course, have linear combinations of more than two vectors. In general, a vector z in E^n is called a linear combination of the vectors $v_1, v_2, ..., v_r$ if it can be expressed in the form

$$z = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r$$

where $\lambda_1, \lambda_2, ..., \lambda_r$ are scalars.

2.5. Linear dependence

The following is another crucial concept from this lecture that you must make sure you understand before next week.
We say that a set of r vectors $\{v_1, v_2, ..., v_r\}$ (all of which are of the same dimension) is linearly dependent if we can find scalars $\lambda_1, \lambda_2, ..., \lambda_r$ (not all zero) such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r = (0, 0, ..., 0) \qquad (1)$$

If the set of vectors $\{v_1, v_2, ..., v_r\}$ is not linearly dependent, then it is linearly independent.
To illustrate, consider the set $\{(2, 1), (-1, 4)\}$ which contains two vectors in E^2. For this set to be linearly dependent, there must exist two scalars λ_1 and λ_2 (at least one of which is not zero) such that

$$\lambda_1(2, 1) + \lambda_2(-1, 4) = (0, 0) \qquad (2)$$

Carrying out the scalar multiplication and then the addition on the left hand side, we can rewrite equation (2) as

$$(2\lambda_1 - \lambda_2, \lambda_1 + 4\lambda_2) = (0, 0) \qquad (3)$$

By comparing the elements of the vectors on the left hand side and the right hand side of (3), we see that equation (3) implies the following pair of simultaneous equations:

$$2\lambda_1 - \lambda_2 = 0 \qquad (4)$$

$$\lambda_1 + 4\lambda_2 = 0 \qquad (5)$$

We can use this pair of equations to 'deduce' whether λ_1 and/or λ_2 can be nonzero as follows: equation (4) implies that $\lambda_1 = \lambda_2/2$. But equation (5) implies that $\lambda_1 = -4\lambda_2$. These two expressions for λ_1 cannot both be true unless $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus, we deduce that the set $\{(2, 1), (-1, 4)\}$ is linearly independent.
This procedure of deriving an equation system from a vector equation like (2) can be used in more complicated cases. Consider, for example, the set

$$\{(2, 3, 4), (0, 1, 2), (4, 3, 2)\}$$

which contains three vectors from E^3. Is this set linearly dependent ? To answer this, let us write out the analogue of equation (2):

$$\lambda_1(2, 3, 4) + \lambda_2(0, 1, 2) + \lambda_3(4, 3, 2) = (0, 0, 0) \qquad (6)$$

As before, we can carry out the scalar multiplications and then the vector additions on the left hand side of equation (6) to get

$$(2\lambda_1 + 4\lambda_3, 3\lambda_1 + \lambda_2 + 3\lambda_3, 4\lambda_1 + 2\lambda_2 + 2\lambda_3) = (0, 0, 0) \qquad (7)$$

By comparing the elements of the three-dimensional vectors on the left hand side and the right hand side of (7), we see that (7) implies the following three-equation system:

$$2\lambda_1 \quad + 4\lambda_3 = 0 \qquad (8)$$

$$3\lambda_1 + \lambda_2 + 3\lambda_3 = 0 \qquad (9)$$

$$4\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0 \qquad (10)$$

From (8), we get that

$$\lambda_1 = -2\lambda_3 \qquad (11)$$

Substituting this result into (9) and rearranging gives

$$\lambda_2 = 3\lambda_3 \qquad (12)$$

Substituting (11) and (12) into (10), we find that (10) is satisfied exactly (check this!). This means that any set of weights of the form $\{-2\lambda_3, 3\lambda_3, \lambda_3\}$ where λ_3 is an arbitrary number can be used to weight the three vectors in the set {(2, 3, 4), (0, 1, 2), (4, 3, 2)} so that their weighted sum is equal to the null vector. For example, setting $\lambda_3 = 1$, we can write

$$-2(2, 3, 4) + 3(0, 1, 2) + (4, 3, 2) = (0, 0, 0)$$

Hence, the set {(2, 3, 4), (0, 1, 2), (4, 3, 2)} must be linearly dependent.

A final thing to note is that linear dependence or linear independence is a property of sets of vectors, not of the vectors themselves, so that the statement '$x = (x_1, x_2, x_3)$ is linearly independent', for example, has no meaning.

2.6. Inner products and orthogonality

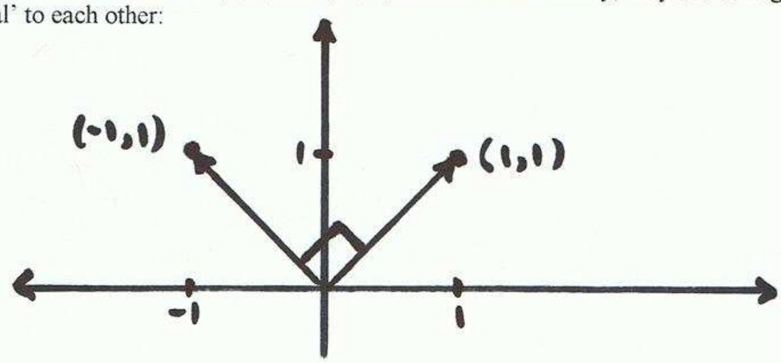
So far, we have not considered the operation of multiplying two vectors together. Once we have defined this operation, we can then give precise algebraic definitions to two geometric concepts we met earlier: that of the length of a vector, and that of two vectors being at 'right angles' to each other (when two vectors are at right angles to each other, we say they are orthogonal).

Given two vectors x and y with n elements each, say $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, their inner product, written $x \cdot y$, is defined as

$$x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \tag{13}$$

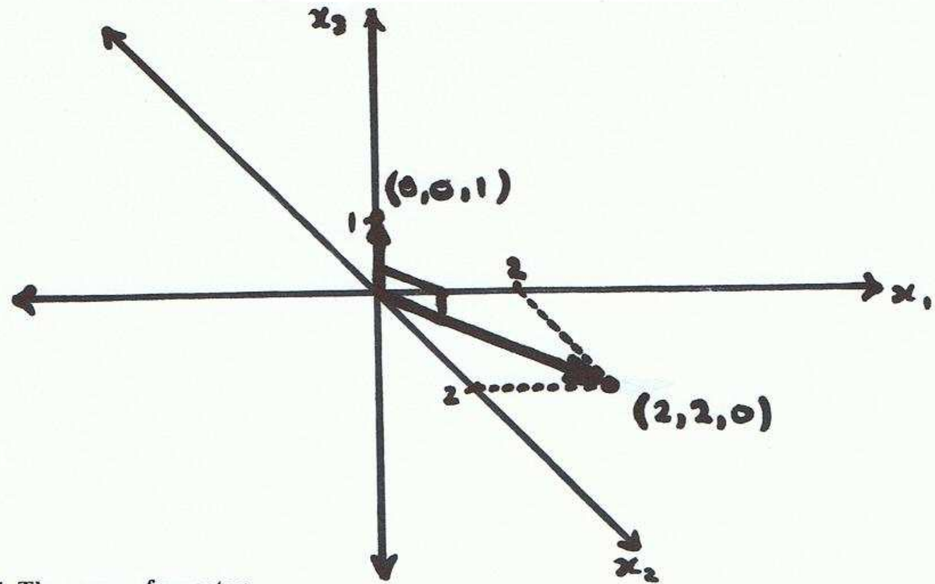
In other words, the inner product of two vectors x and y is the scalar obtained by summing together the products of the corresponding elements of these vectors. For example, if $x = (5, 4)$ and $y = (2, -3)$, then $x \cdot y = (5)\cdot(2) + (4)\cdot(-3) = 10 - 12 = -2$. As another example, if $x = (3, 4, 6)$ and $y = (1, 1, 1)$, then $x \cdot y = 3 + 4 + 6 = 13$.

Now consider the two vectors (1, 1) and (-1, 1) in E^2. Geometrically, they are at 'right angles' or 'orthogonal' to each other:



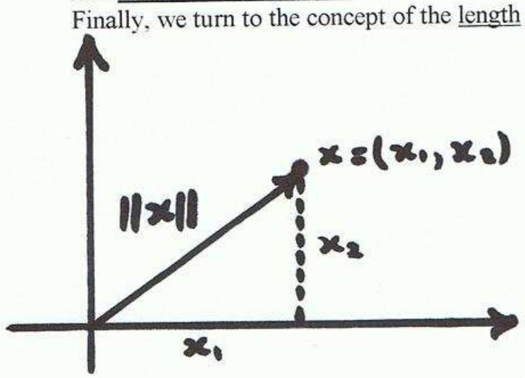
This property is reflected by the fact that the inner product of the two vectors is equal to zero (as you can easily check for yourself). Any two vectors x and y such that $x \cdot y = 0$ are said to be orthogonal to each other, even if we cannot actually visualise them as being at right angles.

For a final example of orthogonality which we can visualise, consider the two vectors (2, 2, 0) and (0, 0, 1) in E^3. These are orthogonal since their inner product is zero (check this!). Geometrically, they are also at right angles in 3-space, as the following diagram shows:



2.7. The norm of a vector

Finally, we turn to the concept of the length of a vector.



By Pythagoras' Theorem:

$$||x|| = \sqrt{x_1^2 + x_2^2}$$

The length of a vector $x = (x_1, x_2)$ is called the norm of x, and is denoted by $||x||$. It follows from Pythagoras' theorem that the norm of a vector $x = (x_1, x_2)$ in 2-space is $||x|| = \sqrt{(x_1^2 + x_2^2)}$. Similarly, if $x = (x_1, x_2, x_3)$ is a vector in 3-space, then $||x|| = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$. We can generalise this concept of length to vectors in E^n by associating with each vector x in E^n a real number $||x||$ which is its norm, defined by

$$||x|| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

Using our inner product notation, this can be written more compactly as $||x|| = \sqrt{(x \cdot x)}$.

Examples

$||(3, 4)|| = \sqrt{(3^2 + 4^2)} = \sqrt{25} = 5$

$||(1, -1, -1)|| = \sqrt{(1^2 + (-1)^2 + (-1)^2)} = \sqrt{3}$

Using the norm $||x||$ of x, one can normalise x by multiplying each of its components by $1/||x||$. The vector thus obtained will have length 1, and is referred to as the normalised vector. For example, $||(3, 4)|| = 5$, so multiplying (3, 4) by 1/5 gives the normalised vector (3/5, 4/5). The norm of (3/5, 4/5) is 1, as you can easily check for yourself.

2.8. What you must do before the next lecture (Thursday, 23rd October, 'Introduction to matrix algebra')

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter Four (pages 67-75), and make sure you are familiar with all the terms and concepts in that Chapter and in these lecture notes. Try some of the problems in the 'Vectors' section in Chapter Four of Chiang to test your understanding. Remember: just like piano practice eventually makes you good at playing the piano, the more problems you do in maths, the more skillful you will become at 'problem solving' (which is an art!). You cannot 'learn' to solve problems by watching me do it on the board.

There will be another short written test at the start of the class next week (Monday 20th October or Tuesday 21st October, depending on which class group you are in) and you will be asked to hand in your answers for assessment. The test will contain questions on the material of Lecture 1, on Supplementary Lecture 1, and on Lecture 2, and is intended as further preparation for the exam on November 6th.

If, after reading the lecture notes and the relevant sections from Chiang, you are still having difficulties with the material, come and see me at once, and I will talk you through it until you understand. Alternatively, see Sara Soares-Carneiro on Tuesdays between 3pm and 4pm in the Portakabin for remedial help. If your performance was poor in the first class test this week, I expect to see a marked improvement in the second one next week. You will probably be asked similar questions to check that you have learnt from past mistakes. The only possible reason for not doing well is that you have not done the reading and/or the problems I have asked you to do, and you have not asked me or Sara for help. The four in-class exams I have to give you during the year will be at least as hard as the short written tests I will give you in each class from now on (and probably somewhat harder), so if you are repeatedly performing poorly in the short tests, you should be very worried indeed!

In the remainder of the class next week, we will go over the problems on the attached assignment sheet for this lecture. Note that this also contains questions relating to Supplementary Lecture 1 on equilibrium analysis. You will be asked to hand in your written solutions at the start of the class next week. These will be marked and returned to you as soon as possible with comments and advice. Again, please make sure you hand in your assignments on time each week. QMII is not like the other courses you are following this year, and you simply cannot afford to put off work until a later date. I will become extremely worried about you if you do not submit written work every single week, and if necessary I will take steps personally to make sure that you are getting on with what I need you to do.

End of Lecture 2