EC2203 QUANTITATIVE METHODS IN ECONOMICS II (1997-1998)

Lecture 3. Introduction to matrix algebra

3.1. Backround

Matrix algebra provides a compact way of writing an equation system. It also provides a way testing for the existence of a solution (via the evaluation of a determinant) and gives a method of finding that solution. However, matrix algebra is applicable only to linear equation systems. Consider the pair of simultaneous equations

$$6x_1 + 3x_2 = 22$$
$$x_1 + 4x_2 = 12$$

This is a linear equation system because the variables x_1 and x_2 do not appear as powers (eg. x_1^2 or x_2^3 etc.), and the variables are not multiplied together in any equation (ie. there are no terms of the form x_1x_2 in the equations). Apart from the variables, the system contains coefficients (ie. 6, 3, 1 and 4) and constants (ie. 22 and 12). We can arrange the set of coefficients as

$$A = \begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix}$$

and the set of variables as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the set of constants as

$$b = \begin{bmatrix} 22 \\ 12 \end{bmatrix}$$

Each of these three arrays constitutes a matrix. The members of each array or matrix are called the elements of the matrix. Note that the elements of the coefficient matrix are separated by blank spaces (not commas).

As a shorthand device, the array in matrix A can be written as $A = [a_{ij}]$, i, j = 1, 2.

Suppose a matrix A has m rows and n columns. Then it is said to be an (m×n) matrix (or to have dimension (m×n)). Note that the number of rows always comes first. In the example above, A is a (2×2) matrix, x is a (2×1) matrix and b is a (2×1) matrix.

A General System of Equations

A general system of m linear equations of the form

$$\begin{matrix} a_{11}x_1 & + a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m \end{matrix}$$

can be arranged as $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(see below). And note that the array in matrix A can be written more compactly as $A = [a_{ij}]$, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$.

Matrices With Special Dimensions

Matrices of certain dimensions are of particular importance. When m=n, the matrix is a square matrix because it has an equal number of rows and columns. If n=1 (ie. only one column) the matrix is called a column vector. If m=1 (ie. only one row) the matrix is called a row vector.

Notice that vectors of the form $x = (x_1, x_2, ..., x_n)$ in E^n can be interpreted as $(1\times n)$ matrices, and all the operations on vectors (addition, scalar multiplication, inner products) are special cases of these operations on more general classes of matrices. If m=n=1, then the matrix has just one element and the matrix is called a scalar. Row\column vectors and scalars are usually represented by a lower case (small) letter, while other matrices are represented by an upper case (capital) letter.

A matrix every element of which is zero is called a null matrix. For example, a null matrix of order (3×2) is written as

$$\underset{(3\times 2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A square matrix D whose elements $d_{ij}=0$ for $i \neq j$ is called a diagonal matrix and is written as

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

for a 3 by 3 matrix.

A diagonal matrix I of order n, where all the elements that lie on the diagonal are equal to 1, is written as

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

This is a particularly important matrix called the identity matrix.

The Transpose of a Matrix

Suppose A is a matrix of order (3×2). The transpose of this matrix, denoted by A', is obtained by interchanging the rows and columns of the original matrix A. For example, if

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}$$

then A' is of order (2×3) and is written as

$$A' = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$$

Note that the elements of A have been changed such that a_{12} becomes a_{21}, a_{21} becomes a_{12}, a_{31} becomes a_{13} and so on....

Note also that the transpose of a column vector is a row vector, and vice versa. For example, take

$$b = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Then b' = [3 1 4]. You could also write the row vector as b' = (3, 1, 4) which is the notation we have been using for vectors up to now.
Finally, note three properties of the transpose:
(1). (A')' = A;
(2). (A + B)' = A' + B';
(3). (AB)' = B'A' if A and B are conformable (see below to find out what that means !)

Symmetric Matrices
A symmetric matrix is a square matrix which is equal to its transpose eg.

$$G = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 1 \\ 4 & 1 & 9 \end{bmatrix} \qquad G' = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

Notice that the elements either side of the leading diagonal are mirror images of each other, with the leading diagonal acting as the mirror.
The sum of two symmetric matrices is always symmetric, but the product need not be, as we shall see.

3.2. Addition and subtraction of matrices
Two matrices can be added (or subtracted) if and only if they have the same dimension. This is called conformability of addition (or subtraction).

Rule: Addition is defined by adding corresponding elements.
Rule: Subtraction is defined by subtracting corresponding elements.

Example1: $G = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix}$

then $G + H = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 4 & 7 \end{bmatrix}$

and $G - H = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 10 & 5 \end{bmatrix}$

Example 2: $G = \begin{bmatrix} 3x & 4x+1 \\ 6x & 2 \end{bmatrix}$ and $H = \begin{bmatrix} 4 & x+2 \\ 3 & 9x \end{bmatrix}$

then $G + H = \begin{bmatrix} 3x & 4x+1 \\ 6x & 2 \end{bmatrix} + \begin{bmatrix} 4 & x+2 \\ 3 & 9x \end{bmatrix} = \begin{bmatrix} 3x+4 & 5x+3 \\ 6x+3 & 9x+2 \end{bmatrix}$

and $G - H = \begin{bmatrix} 3x & 4x+1 \\ 6x & 2 \end{bmatrix} - \begin{bmatrix} 4 & x+2 \\ 3 & 9x \end{bmatrix} = \begin{bmatrix} 3x-4 & 3x-1 \\ 6x-3 & 2-9x \end{bmatrix}$

3.3. <u>Multiplication and division of a matrix by a scalar</u>
<u>Rule</u>: Multiply or divide every element of the matrix by the scalar as appropriate (there is no conformability requirement).

<u>Example</u>: If $G = \begin{bmatrix} 3x+6 & 9 \\ 12 & 6x \end{bmatrix}$ and $c = 3$

then $cG = \begin{bmatrix} 9x+18 & 27 \\ 36 & 18x \end{bmatrix}$ and $\frac{1}{c}G = \begin{bmatrix} x+2 & 3 \\ 4 & 2x \end{bmatrix}$

3.4. <u>Multiplication of one matrix by another</u>
The following conformability condition must be satisfied:

<u>The conformability condition of multiplication</u>
If G is an (m×n) matrix and H is a (p×q) matrix, then the product GH exists if and only if n=p, that is, iff the number of <u>columns</u> n in the first matrix G equals the number of <u>rows</u> p in the second matrix H.

<u>Example 1</u>: Consider the multiplication of a row vector x by a column vector y. Assume that the row vector x is of dimension (1×n) whereas the column vector is of dimension (n×1), so they satisfy the conformability condition above.

$$xy = [x_1\ x_2\ ...\ x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [x_1y_1 + x_2y_2 + \cdots + x_ny_n]$$

The result is a (1×1) matrix, which is just a scalar. For example, multiplication of a specific x and y is

$$xy = [2\ \ 5\ \ -2] \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = [2(3) + 5(-3) + (-2)4] = -17$$

Notice that the dimension of the product matrix in this example is just (1×1). This leads us to the next rule:

<u>Rule</u>: If G is an (m×n) matrix and H is a (p×q) matrix, and if the product GH exists (ie. if n=p), then the dimension of the product matrix GH will be (m×q).

<u>Example 2</u>: $G = [2\ \ 1\ \ 3]$ and $H = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$

The first matrix is (1×3) and the second matrix is (3×2), so the conformability condition is satisfied. Furthermore, the dimension of the product matrix will be (1×2). This (1×2) matrix is computed as follows:

$$GH = [(2\times4)+(1\times3)+(3\times2) \quad (2\times5)+(1\times1)+(3\times1)] = [17 \quad 14]$$

Notice that the process of multiplying a row by a column, term by term, and then adding, is just forming the inner product of the row and the column.

Rule: In general, the (i, k)th element of the product GH (ie. the element in the ith row and kth column of GH) is the inner product of the ith row of G (a row vector) and the kth column of H (a column vector).

Example 3: $G = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

Since G is of dimension (3×2) and H is (2×1), they are conformable for multiplication. The resulting product matrix will be (3×1):

$$GH = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} (1\times5)+(3\times9) \\ (2\times5)+(8\times9) \\ (4\times5)+(0\times9) \end{bmatrix} = \begin{bmatrix} 5+27 \\ 10+72 \\ 20+0 \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Note that the $(1, 1)^{th}$ element of GH equals 32 and is the inner product of the first row of G (ie. [1 3]) and the first (and only) column of H (ie. $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$).

Likewise, the $(2, 1)^{th}$ element of GH equals 82 and is the inner product of the second row of G (ie. [2 8]) and the first (and only) column of H. The $(3, 1)^{th}$ element of GH, 20, is the inner product of the third row of G (ie. [4 0]) and the first (and only) column of H.

Note that in both examples 1 and 2, the 'reverse' product does not exist ie. consider again example 2 where G = [2 1 3] and $H = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$. Suppose we try to find the product matrix HG. The conformability condition for multiplication is not met, since H is (3×2) and G is (1×3). So although we can find the product matrix GH, the product HG does not exist. Only if the matrices are both square is the conformability condition met for either of the products GH or HG.

Example 4: $G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

These matrices are each of order (2×2), and are therefore conformable for multiplication as the products GH and HG. Thus

$$GH = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (1\times1)+(2\times2) & (1\times1)+(2\times0) \\ (3\times1)+(4\times2) & (3\times1)+(4\times0) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 11 & 3 \end{bmatrix}$$

and

$$HG = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (1\times 1)+(1\times 3) & (1\times 2)+(1\times 4) \\ (2\times 1)+(0\times 3) & (2\times 2)+(0\times 4) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$$

Thus, in this example and in most cases, matrix multiplication is not commutative (see Chiang, Chapter Four, pages 76-78). In other words, it is generally the case that GH≠HG.

Example 5: The matrix product GI is commutative, where I is an identity matrix of the same order as G. For example,

$$GI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (a_{11}\times 1)+(a_{12}\times 0) & (a_{11}\times 0)+(a_{12}\times 1) \\ (a_{21}\times 1)+(a_{22}\times 0) & (a_{21}\times 0)+(a_{22}\times 1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

You can now easily verify for yourselves that IG = GI.

Example 6: Recall we noted earlier that while the sum of two symmetric matrices is always symmetric, the product need not be. Here is an example to help fix ideas:

$$A = \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$$

We have

$$AB = \begin{bmatrix} (0\times 1)+(3\times 2) & (0\times 2)+(3\times 7) \\ (3\times 1)+(-2\times 2) & (3\times 2)+(-2\times 7) \end{bmatrix} = \begin{bmatrix} 6 & 21 \\ -1 & -8 \end{bmatrix}$$

which is not symmetric. Thus, AB≠(AB)'.

As a final note on matrix multiplication, observe that because marix multiplication is in general not commutative, we need to distinguish very carefully between pre-multiplication and post-multiplication. In forming the matrix product GH, we say we have post-multiplied G by H, whereas in forming the matrix product HG, we say we have pre-multiplied G by H.

3.5. The inverse of a matrix

The final ingredient of matrix algebra for the purposes of this lecture is 'division'. We view division as multiplication by an inverse. In the case of real numbers, dividing by 3, for example, is equivalent to multiplying by 1/3 or 3^{-1}, the inverse of 3. Also, the inverse of 3 has the property that $3\times 3^{-1} = 1$.

To extend this to matrix algebra, we need some analogue of the number 1. This is provided by the identity matrix, which you will recall is a square matrix with ones along the main diagonal, and zeros everywhere else.

***Definition*: Formally, A^{-1} is the inverse of A if and only if A^{-1} satisfies $A^{-1}A = AA^{-1} = I$.**

Example: If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ because $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note that for a matrix to have an inverse, it must be square, but not all square matrices have an inverse! (Squareness is a necessary but not sufficient condition for a matrix to have an inverse). Square matrices which do have an inverse are called non-singular or invertible. Square matrices which do not have an inverse are called singular.

Some useful facts concerning inverses and transposes

1. If A^{-1} is the inverse of an invertible square matrix A, then A^{-1} is unique ie. there is no other matrix $C \neq A$ such that $C^{-1}A = AC^{-1} = I$.

2. $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are square and non-singular.

3. $(A^{-1})' = (A')^{-1}$.

We saw on page 1 that a general system of m linear equations of the form

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

can be arranged as $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We are now in a position to ask ourselves how this system can be solved. If A^{-1} exists (ie. if A is non-singular), then pre-multiplication of both sides of $Ax = b$ by A^{-1} yields

$$A^{-1}Ax = A^{-1}b$$

or

$$x = A^{-1}b$$

Since A^{-1} is unique <u>if it exists</u>, $A^{-1}b$ must be a unique vector of solution values.

Recall that a <u>necessary condition</u> for the non-singularity of A (ie. for its inverse to exist) is that A must be square. But a square matrix may not have an inverse. A <u>sufficient condition</u> for the non-singularity of A is that its rows are <u>linearly independent</u>. Thus, we will get a unique solution to our equation system Ax = b only if we have linear independence in a square matrix A.

But as you found out in last week's assignment, linear independence cannot always be ascertained at a glance (see the example in Chiang, Chapter Four, page 101 for instance). A test for linear dependence involves evaluating the <u>determinant</u> of the matrix. As we shall see next week, if the determinant of a matrix is zero, there is linear dependence.

3.6. <u>What you must do before the lecture next week (Thursday, 30th October, 'Matrix inversion and Cramer's rule')</u>

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter Four (you should already have read pages 67-75 as part of last week's work), and make sure you are familiar with everything in that chapter and in these lecture notes. Do as many problems as you can from Chapter Four (all of them if possible!) to get accustomed to handling matrices.

There will be another short written test at the start of the class next week (Monday 27th October or Tuesday 28th October, depending on which class group you are in) and you will be asked to hand in your answers for assessment. The test will contain questions on

the material of Lecture 1, Supplementary Lecture 1, Lecture 2 and Lecture 3, and is intended to prepare you for the exam on November 6th.

In the remainder of the class next week, we will go over the problems on the attached assignment sheet for this lecture. You will be asked to hand in your written solutions at the start of the class next week.

End of Lecture 3