EC2203 QUANTITATIVE METHODS IN ECONOMICS II (1997-1998)

Lecture 4. Matrix inversion and Cramer's rule

4.1. Determinants of (2×2) and (3×3) matrices

From Lecture 1, you are familiar with functions which associate a scalar f(x) with a real value of the variable x. They are called scalar-valued functions of one (real) variable. In this section, we shall study the determinant function, which is a specific example of a scalar-valued function of a matrix variable ie. a function associating a scalar f(X) with a matrix X.

The determinant of an (n×n) matrix A is a unique scalar associated with A by a well-defined calculation rule. Determinants are only defined for square matrices.

Let us first consider the determinant of a (2×2) matrix A, where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

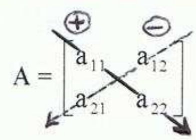
We represent the determinant of A as

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

where the straight lines without the usual corners indicate that it is a determinant. For the (2×2) matrix above, we have

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

In schematic form, the process is illustrated with arrows as



Example: The determinant of the matrix $A = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix}$ is $\det A = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix} = 3(5) - 2(6) = 3$.

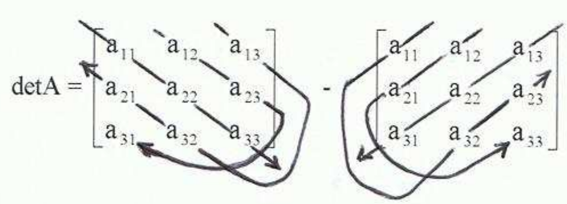
Now consider the determinant of a (3×3) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant for this matrix is computed as

$$\det A = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}] - [a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{32}a_{23}]$$

The derivation of this expression can again be represented schematically by arrows:



Note: The procedures described above are only suitable for square matrices of order 2 or 3.

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4.2. An alternative method for evaluating higher-order determinants: the Laplace expansion

Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then the determinant of A is also given by

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

To understand where this expression comes from, it is necessary to define minors and cofactors.

The subdeterminant formed after deletion of a row and a column as above is called a minor. ***The minor of element a_{ij} (at the intersection of row i and column j) is denoted by $|M_{ij}|$, and is the determinant of the submatrix formed on deletion of the ith row and jth column of the full matrix. It follows from this definition that a matrix has as many minors as it has elements.*** For a (3×3) matrix, we therefore have the following nine minors:

$|M_{11}|=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33} - a_{23}a_{32}$, $|M_{12}|=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{21}a_{33} - a_{23}a_{31}$,

$|M_{13}|=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=a_{21}a_{32} - a_{22}a_{31}$,

$|M_{21}|=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=a_{12}a_{33} - a_{13}a_{32}$, $|M_{22}|=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}=a_{11}a_{33} - a_{13}a_{31}$,

$|M_{23}|=\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}=a_{11}a_{32} - a_{12}a_{31}$,

$|M_{31}|=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}=a_{12}a_{23} - a_{13}a_{22}$, $|M_{32}|=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}=a_{11}a_{23} - a_{13}a_{21}$,

$|M_{33}|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22} - a_{12}a_{21}$.

Thus, the determinant of A above can be written as

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

The cofactor of an element a_{ij}, denoted by $|C_{ij}|$, is the minor of that element prefixed by its correct sign. The sign is given by

$$(-1)^{i+j}$$

where i and j refer respectively to the row and column that have been deleted. The number $(-1)^{i+j}$ is positive if i+j is even, and negative if i+j is odd. So $|C_{ij}| \equiv (-1)^{i+j}|M_{ij}|$. Hence, |A| can be written as

$$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|$$

where

$$|C_{11}| = (-1)^2|M_{11}| = |M_{11}|, \quad |C_{12}| = (-1)^3|M_{12}| = -|M_{12}|, \quad |C_{13}| = (-1)^4|M_{13}| = |M_{13}|$$

The determinant of any square matrix can be found by using this Laplace expansion, which for an (n×n) matrix is

$$|A| = \sum_{j=1}^{n} a_{kj}|C_{kj}|, \text{ for expansion by the kth row.}$$

Note: Since any row or column can be chosen for the Laplace expansion, always choose the row/column with the most zeros or ones!

Summary of Laplace expansion procedure

1). Choose any row or column in the matrix (say column 1).

2). Form the product of the first element in that row or column (here a_{11}) with the minor obtained by deleting that row and column from the matrix (here $|M_{11}|$).

3). If the sum of the row and column index is even, multiply the product by +1. If odd, multiply by -1 (for $a_{11}|M_{11}|$, the sum of the row and column index is 1+1=2 which is obviously even, so the sign of $a_{11}|M_{11}|$ stays unchanged).

4). Repeat steps (2) and (3) for the second element in the row or column along which you are expanding, and add to the first product, and so on, until you come to the end of the row or column (in this example, we would end up with $a_{11}|M_{11}| - a_{21}|M_{21}| + a_{31}|M_{31}| - \cdots + (-1)^{n+1}a_{n1}|M_{n1}|$).

The important thing to notice is that, regardless of which row or column we choose to expand along, we get the same answer. So choose a row or column with lots of zeros.

Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Expanding along the second row we get

$$|A| = a_{21}|C_{21}| + a_{22}|C_{22}| + a_{23}|C_{23}| = -a_{21}|M_{21}| + a_{22}|M_{22}| - a_{23}|M_{23}|$$

$$= -3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -3(2-9) + 2(1-6) - 0 = -3(-7) + 2(-5) = 21 - 10 = 11$$

Example 2

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & -13 \\ -1 & 2 & 17 \end{bmatrix}$$

Expanding along the first row we get

$$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|$$

$$= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

$$= 0\begin{vmatrix} -2 & -13 \\ 2 & 17 \end{vmatrix} - 1\begin{vmatrix} 1 & -13 \\ -1 & 17 \end{vmatrix} + 0\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = -1(17-13) = -4$$

Note: All of the above applies to expansion along any row or column.

4.3. Some properties of determinants

(1). Taking the transpose of a matrix does not affect the value of the determinant ie. $|A| = |A'|$.

For example, $\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix} = 9$.

(2). The interchange of any two rows (or two columns) will alter the sign, but not the numerical value, of the determinant. For example, $\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = 9$; interchanging the two rows, $\begin{vmatrix} 5 & 6 \\ 4 & 3 \end{vmatrix} = -9$.

(3). The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold.

(4). The addition (subtraction) of a multiple of any row to (from) another row leaves the value of the determinant unaltered.

(5). If a row (or column) in a matrix contains only zeros, then $|A|=0$.

(6). If one row (or column) is a linear combination of other rows (or columns), then $|A|=0$.

These properties can be useful in simplifying the evaluation of determinants. In particular, property (6) is used in the establishment of singularity (see below). As an example of (6), consider the matrix $A = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$. The second row of this matrix is just twice the first row. It follows from (6) that the determinant must be zero, and this is indeed the case : $|A| = 4(2) - 1(8) = 0$.

4.4. The rank of a matrix

The rank of a matrix A, denoted by rank(A), is the order of the largest non-zero determinant that can be obtained from the elements of A. This definition applies to both square and rectangular matrices. Thus, a non-null matrix A has rank r if at least one minor of order r is different from zero, while all larger minors (ie. minors of order r+1 or larger) are equal to zero.

The rank of a matrix A can be found by starting with the largest determinants of order m, say, and evaluating them to see if one of them is non-zero. If so, rank(A) = m. If all the determinants of order m are equal to zero, we start evaluating determinants of order m-1. Continuing in this fashion, we eventually find the rank r of the matrix, being the order of the largest non-zero determinant.

Example: Find rank(A), where $A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$. First, we find that $|A| = 6(1) - 3(2) = 0$. Thus, rank(A) = 1, since the order of the largest non-zero minor of A is 1. (There are, in this simple example, four non-zero minors of order 1).

Notice that if A is $(n \times m)$ and $n \neq m$, then $\text{rank}(A) \leq \min(n, m)$.

A very important result is the following: ***If A is $(n \times n)$ and rank(A) = n, then A^{-1} must exist.*** This is because it can be shown that the rank of a square matrix is equal to the number of linearly independent rows in the matrix (which is also equal to the number of linearly independent columns). If A is $(n \times n)$ and rank(A) = n, then all the rows (and columns) of A must be linearly independent, which is a sufficient condition for A^{-1} to exist. A square matrix A of dimension $(n \times n)$ will not be invertible (ie. it will be singular) if its rows (and therefore columns) are linearly dependent. In that case, we would find that rank(A) < n.

Note that we now have a set of equivalent criteria for the existence of A^{-1} : ***A^{-1} exists iff $|A| \neq 0$ $\Leftrightarrow$ rank(A)=n $\Leftrightarrow$ the rows (and columns) of A are linearly independent.***

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4.5. A note on homogeneous equation systems

In the last lecture, we said that a general system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be arranged as $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If b is a null vector, so that $Ax = \underset{(m\times 1)}{0}$, then the m linear equations are said to form a homogeneous equation system. Written out in full, a homogeneous system looks as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

(1). Suppose that A is a square and non-singular ($m\times m$) matrix. Then the homogeneous equation system yields only the 'trivial' solution $x = A^{-1} \underset{(m\times 1)}{0} = \underset{(m\times 1)}{0}$.

(2). Suppose now that A is singular. This means that the equations in the homogeneous system are linearly dependent. It can be shown that in this case, the equation system has an infinite number of non-trivial solutions (in addition to the trivial one).

As an application of these results, consider again problem 6 (e) in the assignment for lecture 2. I asked you to determine whether or not the following set of vectors in 4-space is linearly dependent: {(2, 3, 4, 5), (1, 2, 1, 3), (0, 1, -2, 2), (3, 5, 5, 8)}. As you should know by now, the definition of linear dependence involves the vector equation

$$\lambda_1(2, 3, 4, 5) + \lambda_2(1, 2, 1, 3) + \lambda_3(0, 1, -2, 2) + \lambda_4(3, 5, 5, 8) = (0, 0, 0, 0)$$

This implies the following 4-equation system (I have written it out in matrix form):

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 5 \\ 4 & 1 & -2 & 5 \\ 5 & 3 & 2 & 8 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, this is clearly a homogeneous system. If the coefficient matrix were non-singular, the only possible solution to the system would be the trivial one (ie. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$), in which case the set of vectors would be linearly independent. However, it can be shown that the determinant of the coefficient matrix is equal to zero, so that an inverse does not exist. You can easily see that the fourth column of the coefficient matrix is just the sum of the first two columns. By (6) in section 4.3., this must mean that the determinant of the matrix is zero, and therefore that the coefficient matrix is singular. By (2) above, it follows that the system has an infinite number of non-trivial solutions, so the set of vectors is linearly dependent.

4.6. Evaluation of the inverse A^{-1}

So far, we have discussed the inverse A^{-1} of a matrix without indicating how we calculate it. We now know that, if the matrix A in the linear equation system $Ax = b$ is non-singular, then A^{-1} exists, and the (unique) solution of the system will be $x = A^{-1}b$. We can test for the non-singularity of A by the criterion $|A| \neq 0$. In order to move on to calculating A^{-1}, we must begin by defining some new matrices.

The cofactor matrix

A cofactor matrix C is obtained by replacing each element a_{ij} of a square matrix A by its corresponding cofactor $|C_{ij}|$, where $|C_{ij}|$ is calculated as discussed earlier (see Laplace expansion). Recall that $|C_{ij}| = (-1)^{i+j}|M_{ij}|$, where $|M_{ij}|$ is the minor corresponding to the removal of the ith row and jth column.

Example 1: Find the cofactor matrix for $B = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$.

We have the following : $|C_{11}| = -1$, $|C_{12}| = -4$, $|C_{21}| = -2$, $|C_{22}| = 3$. Therefore the cofactor matrix is $C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -2 & 3 \end{bmatrix}$.

Example 2: Find the cofactor matrix for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

We have the following:

$$|C_{11}| = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 12 - 6 = 6, \; |C_{12}| = -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -(8 - 6) = -2, \; |C_{13}| = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = 6 - 9 = -3,$$

$$|C_{21}| = -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -(8 - 9) = 1, \; |C_{22}| = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 4 - 9 = -5, \; |C_{23}| = -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = -(3 - 6) = 3,$$

$$|C_{31}| = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5, \; |C_{32}| = -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -(2 - 6) = 4, \; |C_{33}| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1.$$

Therefore the cofactor matrix is

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

The adjoint matrix

If we have a square matrix A and its cofactor matrix C, then we define the adjoint matrix of A (written as adjA) as the transpose of the cofactor matrix, so that $\text{adj}A = C'$. Thus, for a (3×3) matrix, if

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix}, \text{ then } \quad \text{adj}A = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{23}| & |C_{33}| \end{bmatrix}.$$

Consider example 2 above. We found that

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

The adjoint of A is simply the transpose of this matrix:

$$\text{adj}A = C' = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

Relationship between the adjoint matrix and finding the inverse A^{-1}

It can easily be demonstrated (see eg. Chiang, Chapter 5, pages 105-107) that

$$\frac{A.\text{adj}A}{|A|} = I$$

where I is the identity matrix. Pre-multiply both sides of this equation by A^{-1}, and recall that $A^{-1}A = I$. This yields the formula for calculating the inverse of a matrix:

$$\frac{\text{adj}A}{|A|} = A^{-1}$$

Example 3: Find the inverse of the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$. First, check the determinant: $|A| = 4-2 = 2$. Since $|A| \neq 0$, the inverse A^{-1} exists. Next we find the cofactor matrix. We have $|C_{11}| = 1$, $|C_{12}| = -2$, $|C_{21}| = -1$, $|C_{22}| = 4$. Therefore the cofactor matrix is $C = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$. Next, we find the adjoint of A. This is just $\text{adj}A = C' = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$. Then $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$.

To check the inverse above, now find $AA^{-1} = I$:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4(1/2)+1(-1) & 4(-1/2)+1(2) \\ 2(1/2)+1(-1) & 2(-1/2)+1(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 4: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$. First, calculate the determinant:

$|A| = [(2)(2)(2)+(3)(3)(3)+(1)(1)(1)] - [(1)(2)(3)+(3)(1)(2)+(2)(1)(3)] = 36 - 18 = 18$. Since $|A| \neq 0$, the inverse A^{-1} exists. Now, we find the cofactor matrix. Verify for yourself that this is

$C = \begin{bmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{bmatrix}$. It follows that $\text{adj}A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$. Therefore

$$A^{-1} = \frac{\text{adj}A}{|A|} = \begin{bmatrix} 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \\ -5/18 & 7/18 & 1/18 \end{bmatrix}$$. You can verify for yourself that $AA^{-1} = I$.

4.7. Cramer's rule

We are now going to state Cramer's rule, which gives a solution value of a single endogenous variable x_i at a time. For a more complete derivation, see Chiang, Chapter 5, pages 107-112. Cramer's rule can be very useful if one only needs to find one variable's value.

Suppose we have a linear equation system $Ax = b$. We have found that

$$x = A^{-1}b = \frac{\text{adj}A}{|A|}b$$

provided that A is non-singular. If we only wanted to find one element of the vector x, say x_i, it would be a waste of effort to go through the whole process of inverting A and then multiplying b by the inverse. Instead we could obtain x_i simply by replacing the ith column of the matrix A by the constant vector b (to get a matrix denoted by B_i), and then using the following simple formula:

$$x_i = \frac{|B_i|}{|A|}$$

This is Cramer's rule for the solution of an equation system $Ax = b$ one variable at a time. Note that Cramer's rule is based on the concept of the inverse matrix, although in practice it avoids the process of matrix inversion.

Example: Find x_1 and x_2 using Cramer's rule, where

$$\begin{bmatrix} 6 & -3 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 35 \end{bmatrix}.$$

First, calculate the determinant of the coefficient matrix: $|A| = 36-6 = 30$. Then using Cramer's rule, we have

$$x_1 = \frac{|B_1|}{|A|} = \frac{\begin{vmatrix} 50 & -3 \\ 35 & 6 \end{vmatrix}}{30} = (300+105)/30 = 405/30 = 13.5$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{\begin{vmatrix} 6 & 50 \\ -2 & 35 \end{vmatrix}}{30} = (210+100)/30 = 310/30 = 10.3$$

4.8. What you must do before the exam next week (Thursday 6th November)

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter 5, Sections 5.1, 5.2, 5.3, 5.4, and 5.5. Learn how to invert a (3×3) matrix by finding its determinant, then the cofactor matrix, then the adjoint matrix, and then dividing each element of the adjoint matrix by the determinant. Please do lots of practice, because you will definitely be asked to invert a (3×3) matrix in the exam. There will not be a test in the class next week. Instead, I will give you a final briefing for the exam, and deal with any outstanding problems.

Please make sure you can solve the problems on the attached assignment sheet for Lecture 4 by the start of next week. You will be asked to solve very similar problems in the exam.

(End of Lecture 4)