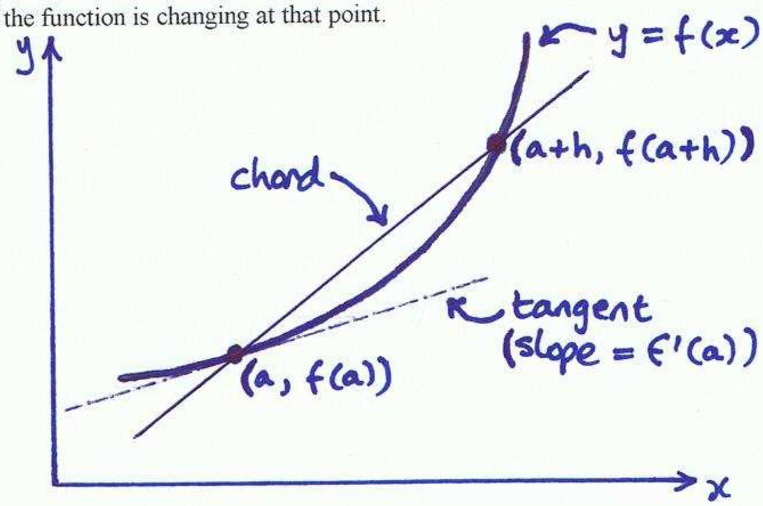
Everything you need to know about differentiation and derivatives for QMII

Introduction

In QMII, it is assumed that you are familiar with the rules of differentiation for functions of a single variable. However, you might find it useful to have a short set of notes which basically covers everything you need to know for this course. Chiang goes into too much detail for my purposes, although you should read him if you can.

The process of differentiation of a function of a single variable y = f(x) is simply the process of finding the slope of a tangent to its graph at a particular point, say (a, f(a)). We call the slope of this tangent 'the derivative of y = f(x) at (a, f(a))'. This derivative measures the rate at which the value of the function is changing at that point.



To find the slope of the tangent to the graph of y = f(x) at the point (a, f(a)), we can follow the following procedure:

1). Write down the coordinates of the point on the graph whose x coordinate is a+h, where h is some number. The coordinates of this point must be of the form (a+h, f(a+h)).
2). Write down the slope of the chord joining (a, f(a)) to (a+h, f(a+h)). This slope is

$$\frac{f(a+h) - f(a)}{a+h - a} = \frac{f(a+h) - f(a)}{h}$$

3). Investigate what happens to this chord slope as h is made close to zero.

The process in Step 3 is referred to as 'letting h tend to zero'. Thus, in Step 3, the question is whether

$$\frac{f(a+h) - f(a)}{h}$$

tends to some value as h tends to zero. If so, then the value obtained is the slope of the tangent to the graph at (a, f(a)), which is what we mean by 'the derivative of y = f(x) at the point (a, f(a))'. The following is a nice example which will help to fix these ideas:

Example: Find the slope of the tangent to the graph of $y = f(x) = x^2$ at the point (a, a^2).
Solution: We proceed step-by-step as follows:

Step 1. The coordinates of (a+h, f(a+h)) are $(a+h, (a+h)^2) = (a+h, a^2+2ah+h^2)$.
Step 2. The slope of the chord joining (a, a^2) to $(a+h, a^2+2ah+h^2)$ is

$$\frac{f(a+h) - f(a)}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h$$

Step 3. As h tends to zero, this chord slope tends to 2a. So the slope of the tangent at (a, a^2) on the graph of $y = f(x) = x^2$ is 2a.

The result in the above example can be summed up as follows: ***for $y = f(x) = x^2$, the slope of the tangent = twice the x-coordinate of the point***. So for each possible x-value a in E^1, there is an associated slope 2a for the tangent of the graph at the point (a, a^2). In other words, we have defined a 'slope function' mapping each possible value of a to the slope of the tangent at x = a. The standard name for the 'slope function' is the *derived function*, and there is also a standard notation: the derived function of a function y = f(x) is denoted by dy/dx or df(x)/dx or f'(x). Thus, in our example above, if $y = f(x) = x^2$, then the derived function is written as

$$\frac{dy}{dx} = 2x \quad \text{or} \quad \frac{df(x)}{dx} = 2x \quad \text{or} \quad f'(x) = 2x$$

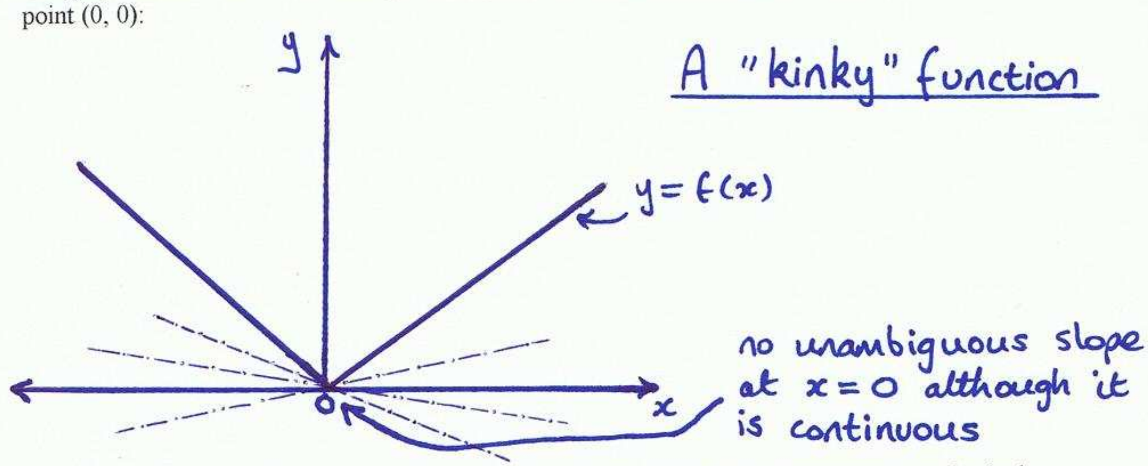
Remember that the technique for finding derived functions is to look at the value of

$$\frac{f(a+h) - f(a)}{h}$$

as h tends to zero. We call this value 'the limit as h tends to zero', and define the derivative function in general terms as

$$\frac{df(x)}{dx} \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

You will almost certainly have come across this definition before in textbooks. Note that for the limit to exist at a particular point (a, f(a)), it is necessary that the function be continuous at that point. (There is a formal definition of continuity which I do not want to get into here, but basically, a continuous function is one that you can draw 'without taking your pen off the paper'. In other words, the graph has no 'gaps' in it). But continuity is not sufficient, because a function like the one graphed below has no unambiguous slope at x = 0, and is therefore not differentiable at the point (0, 0):



There is an important class of economic problems which cannot be solved by the use of calculus because of a lack of differentiability - eg. because of 'kinks' as illustrated above.

Rules for differentiating various functional forms

The following fundamental rules were discovered by applying the procedures described above. Note that the derivatives are stated for general values of x.

Rule (1). If $y = f(x) = c$, where c is a constant, then

$$\frac{df(x)}{dx} = 0$$

Obviously, if f(x) is a constant, in other words is independent of x, then the rate of change of f(x) with respect to x is everywhere zero.

Rule (2). If $y = f(x) = x^2$, then as we saw above,

$$\frac{df(x)}{dx} = 2x$$

Rule (3). This is a generalisation of rule (2). If $y = f(x) = x^n$, for any index $n \neq 0$, then

$$\frac{df(x)}{dx} = nx^{n-1}$$

Some examples:

If $y = f(x) = x^7$, then $f'(x) = 7x^6$.
If $y = f(x) = x^{-3}$, then $f'(x) = -3x^{-4}$.
If $y = f(x) = x^{1/4}$, then $f'(x) = \frac{1}{4}x^{-3/4}$.

Rule (4). If $y = f(x) = cg(x)$, where c is a constant and g(x) is another function of x, then

$$\frac{df(x)}{dx} = c\frac{dg(x)}{dx}$$

For example, $d(cx^n)/dx = cnx^{n-1}$.

Rule (5). The derivative of the sum of two functions is the sum of their derivatives: If

$$f(x) = h(x) + g(x)$$

then

$$\frac{df(x)}{dx} = \frac{dh(x)}{dx} + \frac{dg(x)}{dx}$$

For example,

$$f(x) = 2x^2 + 3x^{-2} \Rightarrow f'(x) = 4x - 6x^{-3}$$

Note that generally, if

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$

then

$$f'(x) = b_1 + 2b_2x + \cdots + nb_nx^{n-1}$$

so that we can differentiate 'term by term'.

Rule (6). The product of two functions. Let

$$f(x) = h(x)\cdot g(x)$$

Then

$$\frac{df(x)}{dx} = h(x)\frac{dg(x)}{dx} + g(x)\frac{dh(x)}{dx}$$

ie. 'the first times the derivative of the second, plus the second times the derivative of the first'. For example, let $f(x) = x^3 \cdot x^4$. This can be differentiated either by rule (3) or rule (6). By rule (3):

$$f(x) = x^7 \text{ so } f'(x) = 7x^6$$

By rule (6): we let $h(x) = x^3$ and $g(x) = x^4$. Then $f'(x) = x^3 4x^3 + x^4 3x^2 = 4x^6 + 3x^6 = 7x^6$, as before.

Rule (7). The 'chain rule'. This rule tells us how to deal with a common situation where, say, some variable y depends on z, and z depends on x, and we wish to find the rate of change of y with respect to changes in x. For instance, if

$$y = g(z) \quad \text{and} \quad z = h(x), \quad \text{so} \quad y = g(h(x))$$

we have that z, the argument of g, is itself a function, so that y 'is a function of a function'. If $y = f(x) = g(h(x))$, then the chain rule says

$$\frac{df(x)}{dx} = \frac{dg(h)}{dh} \cdot \frac{dh(x)}{dx}$$

For example, suppose that $f(x) = (3x+2)^3$. Then take $h(x) = 3x+2$, and $g(h(x)) = h^3$. Using the chain rule,

$$\frac{df(x)}{dx} = \frac{dg(h)}{dh} \cdot \frac{dh(x)}{dx} = 3h^2 \cdot 3 = 9h^2 = 9(3x+2)^2$$

Rule (8). The ratio of two functions. If

$$f(x) = \frac{g(x)}{h(x)}$$

then

$$\frac{df(x)}{dx} = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

For example, let

$$f(x) = \frac{x^2}{x^5}$$

This can be differentiated either by Rule (3) or by Rule (8). Using Rule (3):

$$f(x) = x^{-3}, \text{ so } f'(x) = -3x^{-4}$$

Using Rule (8),

$$f'(x) = \frac{2x \cdot x^5 - x^2 \cdot 5x^4}{[x^5]^2} = \frac{2x^6 - 5x^6}{x^{10}} = -3\frac{x^6}{x^{10}} = -3x^{-4}, \text{ as before.}$$

Also, if

$$f(x) = \frac{(x^2+3)}{(x+2)^3}$$

then

$$f'(x) = \frac{2x \cdot (x+2)^3 - (x^2+3) \cdot 3(x+2)^2}{\left[(x+2)^3\right]^2} = \frac{2x \cdot (x+2)^3 - (x^2+3) \cdot 3(x+2)^2}{(x+2)^6}$$

$$= 2x(x+2)^{-3} - 3(x^2+3)(x+2)^{-4}$$

Rule (9). If $f(x) = \log x$, then $f'(x) = 1/x$.

Rule (10). If $f(x) = e^x$, then $f'(x) = e^x$.

Some problems for practice

Please attempt the following problems. Handwritten solutions are attached.

Differentiate the following functions with respect to x.

(a). $f(x) = x^4$

(b). $f(x) = x^6$

(c). $f(x) = x^{n+1}$

(d). $f(x) = x^{n/m}$

(e). $f(x) = x^{-3/2}$

(f). $f(x) = 6x^5$

(g). $f(x) = \frac{x^{1-a-b}}{(a+b-1)}$

(h). $f(x) = x^5(2x^2+1)$

(i). $f(x) = (x^5+x^2)(x^3+x)$

(j). $f(x) = (x+1)^3$

(k). $f(x) = \frac{(x^2+1)}{x^3}$

(l). $f(x) = \frac{3x+2}{4x^2+3}$

(m). $f(x) = e^{9x}$

(n). $f(x) = x^3e^{9x}$

(End of notes)

Solutions

(a) Using rule (3), $f'(x) = 4x^3$

(b) Using rule (3), $f'(x) = 6x^5$

(c) Using rule (3), $f'(x) = (n+1)x^n$

(d) Using rule (3), $f'(x) = \frac{n}{m} x^{\frac{n-m}{m}}$

(e) Using rule (3), $f'(x) = -\frac{3}{2} x^{-5/2}$

(f) Using rule (3), $f'(x) = 30x^4$

(g) Using rule (3), $f'(x) = -x^{-a-b}$

(h) Using rule (6), $f'(x) = x^5 \cdot 4x + (2x^2+1) \cdot 5x^4$

$$= 14x^6 + 5x^4$$

(i) Using rule (6), $f'(x) = (x^5 + x^2) \cdot (3x^2+1) + (x^3+x) \cdot (5x^4+2x)$

$$= 8x^7 + 6x^5 + 5x^4 + 3x^2$$

(j) Using rule (7), $f'(x) = 3(x+1)^2 \cdot 1 = 3(x+1)^2$

(k) Using rule (8),

$$f'(x) = \frac{2x \cdot x^3 - (x^2+1) \cdot 3x^2}{[x^3]^2} = \frac{2x^4 - 3x^4 - 3x^2}{x^6}$$

$$= -x^{-2} - 3x^{-4}$$

(l) Using rule (8),

$$f'(x) = \frac{3\cdot(4x^2+3)-(3x+2)\cdot 8x}{(4x^2+3)^2}$$

$$= \frac{12x^2+9-24x^2-16x}{(4x^2+3)^2}$$

$$= \frac{-12x^2-16x+9}{(4x^2+3)^2}$$

(m) Using rules (7) and (10): Let $h(x) = 9x$ and $g(h) = e^h$

Then $f'(x) = \frac{dg(h)}{dh}\frac{dh(x)}{dx} = e^h\cdot 9 = 9e^{9x}$

(n) Using rules (6), (7) and (10):

$$f'(x) = x^3(9e^{9x}) + e^{9x}\cdot 3x^2$$

$$= e^{9x}(9x^3+3x^2).$$