# Lecture 5. Partial differentiation

## 5.1. Introduction: functions of several variables

From your previous studies (either at 'A' level or in QMI), you should be familiar with the concept of *differentiation* of functions of a single variable. Since many things are determined by more than one variable, we now wish to generalise the concept of differentiation to cover *multivariate functions* which map points in $E^n$ to points in $E^1$. In other words, we now have

$$f: D \subset E^n \rightarrow R \subset E^1$$

and we write

$$y = f(x_1, x_2, \ldots, x_n).$$

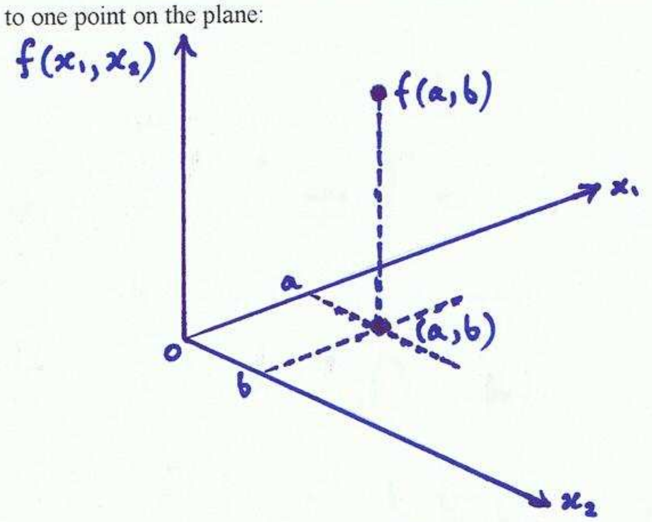
As a typical example from economics, consider a production function which gives the maximum amount of some output y which can be obtained by using amounts $x_1$ and $x_2$ of two inputs:

$$y = f(x_1, x_2)$$

This might have the Cobb-Douglas form

$$y = f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta} \quad (\alpha, \beta, A \text{ are positive constants}).$$

We saw in lectures 1 and 2 that, for a function of two variables, the value of the function can be plotted on the vertical axis, and values of the arguments on the horizontal plane. The graph of the function will be a *surface* in three-dimensional space, with each point on the surface corresponding to one point on the plane:




For a function from $E^2$ to $E^1$, say $y = f(x_1, x_2)$, we identify two slopes at every point on the surface: the slope when we move parallel to the $x_1$ axis (that is, the rate of change of f with respect to changes in $x_1$, *holding $x_2$ constant*), and the slope when we move parallel to the $x_2$ axis (that is, the rate of change of f with respect to changes in $x_2$, *holding $x_1$ constant*). These two slopes are called the *partial derivatives* of the function. Chiang (Chapter 7, pages 176-177) provides a very nice geometric interpretation of partial derivatives, which you should make every effort to understand. It can be shown that the slope given by moving in any other direction on the three-dimensional surface can be written as a linear combination of these partial derivatives, assuming a 'well-behaved' function (which we always assume in QMII !).

More generally, for a function from $E^n$ to $E^1$, we identify n slopes at every point on the n-dimensional 'hypersurface', these slopes being the n partial derivatives of the function.

## 5.2. Partial derivatives

Recall that in the case of a function of a single variable, say $y = f(x)$, we define the general derivative of the function with respect to the variable x as

$$\frac{df(x)}{dx} \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Given a function of n variables, say $y = f(x_1, x_2, \ldots, x_n)$, we now define the ith *partial derivative* (ie. the partial derivative of f with respect to $x_i$) as

$$\frac{\partial f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{\partial x_i} \equiv \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h}$$

provided that this limit exists. That is, the value of the function when the ith element is changed by some small number h, minus the value of the function at the original point, divided by the change in $x_i$. If you compare this with the definition of the simple derivative, you will see that it is really just a simple derivative, with all the other variables regarded as constants. **To partially differentiate a multivariate function with respect to one of its variables, all you have to do is treat the other (n-1) variables as constants, and simply follow the usual rules for univariate calculus !** Partial differentiation adds nothing new to the simple rules for differentiating functions of a single variable, which you should already know !

Example 1: Consider the following function from $E^2$ to $E^1$: $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$. Find the partial derivatives.

Solution: Since this is a function of two variables, there are two partial derivatives. These are $\partial y/\partial x_1$ and $\partial y/\partial x_2$. To find $\partial y/\partial x_1$, we imagine that $x_2$ is a constant, and then differentiate with respect to $x_1$ as if the function were only a function of $x_1$:

$$\frac{\partial y}{\partial x_1} = 6x_1 + x_2$$

Next, to find $\partial y/\partial x_2$, we imagine that $x_1$ is a constant, and then differentiate with respect to $x_2$ as if the function were only a function of $x_2$:

$$\frac{\partial y}{\partial x_2} = x_1 + 8x_2 .$$

Example 2: Consider the following function from $E^2$ to $E^1$: $y = f(u, v) = (u+4)(3u+v)$. Find the partial derivatives.

Solution: Once again, this is a function of two variables, so there are two partial derivatives. These are $\partial y/\partial u$ and $\partial y/\partial v$. To find $\partial y/\partial u$, we imagine that v is a constant, and then differentiate with respect to u as if the function were only a function of u. Thus, using the product rule, we get

$$\frac{\partial y}{\partial u} = 1 \cdot (3u + v) + (u + 4) \cdot 3 = 6u + v + 12$$

Analogously, to find $\partial y/\partial v$, we imagine that u is a constant, and then differentiate with respect to v as if the function were only a function of v:

$$\frac{\partial y}{\partial v} = u + 4 .$$

It is important when writing derivatives to distinguish between the symbols d and $\partial$. The 'curly' $\partial$ indicates that there are variables being held constant. Note that the ith partial derivative is often written $f_i(x_1, x_2, \ldots, x_n)$, or just $f_i$, the subscript i indicating that the differentiation is with respect to $x_i$. We shall often use this more concise notation.

As an example from economics, consider the Cobb-Douglas production function

$$y = f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$$

Since this is a function of two variables, there are two partial derivatives: $\partial y/\partial x_1$ and $\partial y/\partial x_2$. To find $\partial y/\partial x_1$, we imagine that $x_2$ is a constant, and then differentiate with respect to $x_1$ as if the function were only a function of $x_1$:

$$\frac{\partial y}{\partial x_1} = \alpha A x_1^{\alpha-1} x_2^{\beta}$$

Next, to find $\partial y/\partial x_2$, we imagine that $x_1$ is a constant, and then differentiate with respect to $x_2$ as if the function were only a function of $x_2$:

$$\frac{\partial y}{\partial x_2} = \beta A x_1^{\alpha} x_2^{\beta-1}$$

These partial derivatives are the marginal products of the inputs $x_1$ and $x_2$, which can be approximately interpreted as the extra output produced as a result of increasing one of the inputs by one unit, keeping the other inputs constant (*ceteris paribus*).

As a final example, if

$$y = f(x_1, x_2) = x_1 + 2x_1x_2 + x_2^2$$

then

$$f_1 = 1 + 2x_2$$

and

$$f_2 = 2x_1 + 2x_2.$$

**5.3. Second-order partial derivatives**

Suppose we have the multivariate function $y = f(x_1, x_2, \ldots, x_n)$. Each of the n partial derivatives of this function is itself a function of $x_1, x_2, \ldots, x_n$. We can partially differentiate each partial derivative with respect to each of the variables $x_1, x_2, \ldots, x_n$ to get what are known as the second-order partial derivatives. To fix ideas, consider the following simple example:

Example: Suppose $y = f(u, v) = 3u^2v$. Find the second-order partial derivatives.
Solution: First, we must find the two partial derivatives of this function. We have

$$\frac{\partial y}{\partial u} = 6uv \qquad \text{and} \qquad \frac{\partial y}{\partial v} = 3u^2$$

To get the second-order partial derivatives, we now partially differentiate each of these with respect to u and v. We get

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right) = \frac{\partial^2 y}{\partial u^2} = 6v$$

$$\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right) = \frac{\partial^2 y}{\partial u \partial v} = 6u$$

when we partially differentiate the first partial derivative with respect to u and v, and

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial v \partial u} = 6u$$
$$\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial v^2} = 0$$

when we partially differentiate the second partial derivative with respect to u and v. These four expressions are the second-order partial derivatives of $y = f(u, v) = 3u^2v$. We can use the more concise notation $f_{ij}$ to denote 'the partial derivative of the partial derivative $f_i$ with respect to the jth variable'. Thus, we can rewrite the above expressions more neatly as $f_{11} = 6v$, $f_{12} = 6u$, $f_{21} = 6u$, $f_{22} = 0$.

The second-order partial derivatives $\partial^2 y/\partial u \partial v = f_{12}$ and $\partial^2 y/\partial v \partial u = f_{21}$ are called the *cross-partials* of $y = f(u, v)$. Notice that in the above example, we have $f_{12} = f_{21} = 6u$. **The equality of cross-partials is a general property of differentiable functions. For 'well-behaved' functions, it is always true that $f_{ij} = f_{ji}$. This is known as Young's Theorem.**

Since second-order partial derivatives carry two subscripts, it is natural to consider them as elements of a matrix. For a function $y = f(x_1, x_2, ..., x_n)$, the matrix would look as follows:

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

This matrix of second-order partial derivatives is called the *Hessian matrix* of the function $y = f(x_1, x_2, ..., x_n)$. The Hessian matrix is always square, and always symmetric. It is always symmetric, because the equality of cross-partials $f_{ij}=f_{ji}$ means that the off-diagonal elements are mirror images of each other, with the main diagonal of the matrix acting as the mirror. We shall see later how this matrix is used in considering second-order conditions for maxima and minima.

### 5.4. Comparative statics

In Supplementary Lecture 1, we saw that 'equilibrium' in an economic model is an idealised scenario in which all the endogenous variables (eg. price, quantity demanded and quantity supplied in a market model) are in a 'state of rest'. We used three equations to characterise the demand and supply model:

$$Q_d(P) = a - bP \quad (b > 0)$$
$$Q_s(P) = c + dP \quad (d > 0)$$
$$Q_d = Q_s$$

The three endogenous variables are quantity demanded ($Q_d$), quantity supplied ($Q_s$) and price (P). The parameters of the model are a, b, c and d. We learnt how to solve the model ie. how to find an expression for each endogenous variable in terms of parameters alone. The expression for the equilibrium price was $P^* = (a-c)/(b+d)$, and the expression for the equilibrium quantity was $Q^* = (ad+bc)/(b+d)$.

In comparative static analysis, we examine how the equilibrium values of the endogenous variables in the model are affected by changes in the values of exogenous variables and parameters. We begin with an initial equilibrium state, eg. $P^* = (a-c)/(b+d)$ and $Q^* = (ad+bc)/(b+d)$, and allow a *disequilibriating* change in the model eg. a change in the parameter a. We assume that a new equilibrium state occurs after the change in the parameter, and then consider how this new equilibrium compares with the old one. The analysis is *static* because we are only comparing equilibrium states. We are not interested in *whether* or *how* a new equlibrium arises, only in how the new equilibrium compares with the old, once the new one has been reached.

Comparative static analysis may be either quantitative or qualitative. It will be quantitative if we can predict the numerical magnitude of the change in an equilibrium endogenous variable as a result of a change in an exogenous variable or parameter. It will be qualitative if we can only determine the sign or direction of the change.

In comparative static analysis we are considering the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular exogenous variable or parameter. Hence our concern with derivatives which, you will recall, measure the rate at which a function is changing at a particular point.

**5.5. Application of partial differentiation to comparative static analysis of the market model**

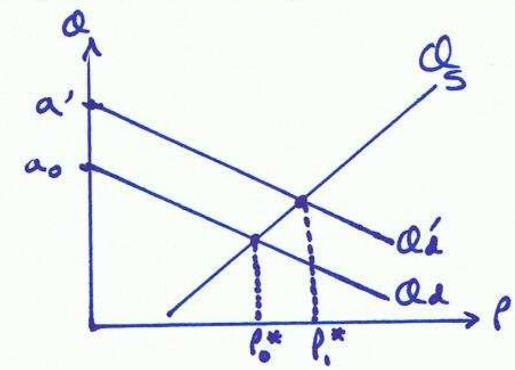
In the following example, the equilibrium values of the endogenous variables can be expressed explicitly in terms of the exogenous variables and parameters. This is called the reduced form of the model. Thus all that is needed is the technique of partial differentiation. We shall learn new techniques later on to deal with the much more common situation in which we do not have explicit expressions for the endogenous variables, only 'implicit' ones.

The reduced form of the market model, as we have just seen, is

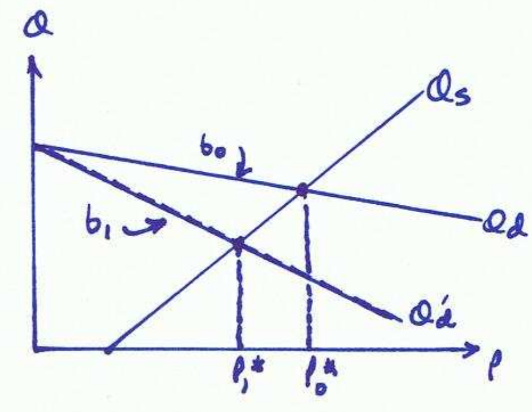
$$P^* = \frac{a-c}{b+d} \qquad Q^* = \frac{ad+bc}{b+d}$$

Remember from Supplementary Lecture 1 that, to have economic meaning, we must impose the restrictions a > c and ad+bc > 0. Now consider two comparative static results:

(i). $\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0$




(ii). $\frac{\partial P^*}{\partial b} = -\frac{(a-c)}{(b+d)^2} < 0$



The first result tells us that an increase in a will lead to an increase in the equilibrium price, and a decrease in a will lead to a decrease in the equilibrium price ie. that equilibrium price is positively correlated with the parameter a. A change in a leads to a change in P* in the same direction. The second result tells us that an increase in b will lead to a decrease in the equilibrium price, and that a decrease in b will lead to an increase in the equilibrium price ie. that equilibrium price is negatively correlated with the parameter b. A change in b leads to a change in P* in the opposite direction. These are typical of the results one gets from comparative static analysis.

Clearly, you can also find $\partial P^*/\partial c$ and $\partial P^*/\partial d$, as well as find the comparative statics of Q*: $\partial Q^*/\partial a$, $\partial Q^*/\partial b$, $\partial Q^*/\partial c$ and $\partial Q^*/\partial d$. You will be asked to find these partial derivatives and interpret them as part of your assignment for this week.

In this very simple example, we were able to show the results graphically. This is not always possible, either because we may be in $E^n$, or because in later examples, we will not always work with explicit forms of demand and supply equations.

### 5.6. What you must do before the lecture next week (Thursday, 27th November, 'Total differentiation')

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter 7, pages 174-182. Make sure you are familiar with all the terms and concepts in there, and in these lecture notes. Exercise set 7.4 in Chiang is extremely useful, and I strongly recommend that you work through it if you possibly can. Please do all the problems on the attached assignment sheet for Lecture 5. Remember: by doing this, you will in all probability be doing your next exam questions in advance! As usual, you will be asked to hand in your written solutions at the start of class next week.

**There will be a short written test at the start of class next week (Monday 24th November or Tuesday 25th November, depending on which class group you are in) and you will be required to hand in your answers for assessment. The test will be on quadratic forms (Supplementary Lecture 2) and on the material of this lecture. You are not expected to do any special revision for this. You should be able to do the simple questions I will ask if you have completed your assignment.** In the remainder of the class next week, we will cover some absolutely essential material prior to the main lecture on Thursday. Further handouts will be distributed, but you will find them difficult to understand unless you watch me setting out the material on the board, so please make sure you attend the class. It will save you (and me!) a lot of time and effort. I am under extreme pressure at the moment, and will not be able to help you if you encounter difficulties because you have not done what I have asked.

Please also remember that your next exam is less than four weeks away.

**(End of Lecture 5)**