A note on Jacobian determinants

From our work on matrix algebra in the first part of this course, we know that a system of n linear functions from E^n to E^1 of the form

$$y_1 = f^1(x_1, x_2, \ldots, x_n) = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$y_2 = f^2(x_1, x_2, \ldots, x_n) = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$y_n = f^n(x_1, x_2, \ldots, x_n) = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

can be arranged in matrix terms as y = Ax, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We learnt how to test for linear dependence by calculating the determinant of A. The n functions in the system are linearly dependent if and only if $|A| = 0$. The determinant $|A|$ is a specific example of a Jacobian determinant, which allows us to test for *functional* dependence (both linear and non-linear). A Jacobian determinant, denoted by $|J|$, is composed of all the first-order partial derivatives of a system of equations, arranged in ordered sequence. It can easily be seen that each element a_{ij} of the matrix A in the linear system above is the partial derivative of the function y_i with respect to the variable x_j. In other words, $\partial y_i/\partial x_j = a_{ij}$. Thus, we can write

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_n/\partial x_1 & \partial y_n/\partial x_2 & \cdots & \partial y_n/\partial x_n \end{vmatrix}$$

Notice that the elements of the ith row of A are the partial derivatives of the ith function with respect to each of the independent variables $x_1, x_2, \ldots, x_n$. Given any system of n differentiable functions in n variables, $|J|$ is the determinant of the matrix containing the n^2 first-order partial derivatives of the functions arranged in this particular order.

In the case of the system of linear functions above, the Jacobian determinant picks up linear dependence, since $|A| = |J| = 0$ when the functions are linearly dependent. More generally, given a system of n non-linear functions in n variables, the Jacobian determinant can pick up functional dependence which is non-linear as well, as the following simple example shows:

Example: Consider the following pair of non-linear functions from E^2 to E^1:

$$y_1 = f(x_1, x_2) = 5x_1 + 3x_2$$
$$y_2 = f(x_1, x_2) = 25x_1^2 + 30x_1x_2 + 9x_2^2$$

The first-order partial derivatives are $\partial y_1/\partial x_1 = 5$, $\partial y_1/\partial x_2 = 3$, $\partial y_2/\partial x_1 = 50x_1 + 30x_2$, $\partial y_2/\partial x_2 = 30x_1 + 18x_2$. Thus the Jacobian determinant is

$$|J| = \begin{vmatrix} 5 & 3 \\ 50x_1 + 30x_2 & 30x_1 + 18x_2 \end{vmatrix} = 5(30x_1+18x_2) - 3(50x_1+30x_2) = 0$$

Since $|J|=0$, there is functional dependence between the equations. It can easily be seen that the second equation is just the square of the first. To complete your coverage of this topic, please read Chiang, Chapter 7, pages 184-186.

(End of note)