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# Lecture 6. Total differentiation

## 6.1. Introduction: from *partial* differentiation to *total* differentiation

Consider a function from $E^n$ to $E^1$, say $y = f(x_1, x_2, ..., x_n)$. The technique of partial differentiation assumes that all the variables $x_1, ..., x_n$ are *independent of each other*, in the sense that a change in any one of the variables has no effect on the others. Thus, we have been able to find $\partial y/\partial x_1$, for example, by assuming that $x_1$ can vary without affecting the values of $x_2, ..., x_n$. The variables $x_2, ..., x_n$ are simply treated as constants when partially differentiating with respect to $x_1$.

We saw how to use partial differentiation to analyse the comparative statics of a simple market model in the last lecture. It was possible to use partial differentiation in this case because we had explicit reduced form solutions for the model ie. we could express P* and Q* explicitly as functions of the parameters a, b, c, and d: $P^*(a, b, c, d) = (a-c)/(b+d)$ and $Q^*(a, b, c, d) = (ad+bc)/(b+d)$. Since a change in any one of the parameters a, b, c, d has no effect on any of the remaining parameters, we were able to find the eight comparative static partial derivatives of the model by partially differentiating P* and Q* with respect to each parameter in turn: $\partial P^*/\partial a$, $\partial P^*/\partial b$, $\partial P^*/\partial c$, $\partial P^*/\partial d$, $\partial Q^*/\partial a$, $\partial Q^*/\partial b$, $\partial Q^*/\partial c$ and $\partial Q^*/\partial d$.

Unfortunately, it is difficult in practice to set up an acceptably 'realistic' model which yields explicit reduced form solutions, because very strong assumptions about functional forms and parameter values have to be made. In the case of the simple market model above, for example, we had to assume that demand and supply are simple linear functions of price. ***When setting up a mathematical model to investigate some economic phenomenon, your aim should always be to keep things as general as possible by imposing as few restrictions as possible on the model.*** In this way, any interesting results you come up with will be applicable in a wide range of circumstances, not just when all functional forms are linear (for example).

In this lecture, we consider what to do when we have no explicit reduced form solution to our model, perhaps because we have not wanted to restrict ourselves to any specific functional forms. In this situation, we do not have neat expressions in terms of exogenous variables and parameters which are all independent of each other, so *partial* differentiation is inappropriate. We have to use total differentiation to investigate the comparative statics of the model.

## 6.2. A simple general-function national income model with two endogenous variables

To fix ideas, consider a simple national income model of the type discussed in Supplementary Lecture 1. The endogenous variables are

$Y \equiv$ national income or output (in £ per period)

$C \equiv$ aggregate household consumption (in £ per period)

and the exogenous variables are

$I \equiv$ investment expenditure by firms (in £ per period)

$G \equiv$ government spending (in £ per period)

$T \equiv$ taxes (in £ per period)

To begin with, let us specify an explicit equation system linking these variables together. The first equation is the equilibrium condition (discussed in Supplementary Lecture 1):

$$Y = C + I_0 + G_0$$

The second equation is the Keynesian consumption function

$$C = a + b(Y - T_0) \qquad (a>0,\ 0<b<1)$$

where $Y-T_0$ is the level of *disposable* income. Following the same general procedure as discussed in Supplementary Lecture 1, we can solve for equilibrium income in terms of exogenous variables and parameters by substituting the (explicit) equation for C into the equilibrium condition:

$$Y = a + b(Y-T_0) + I_0 + G_0$$

Solving this expression for Y gives the reduced form solution for equilibrium income:

$$Y^*(a, b, I_0, G_0, T_0) = (a + I_0 + G_0 - bT_0)/(1-b)$$

We now substitute this expression for Y* into the consumption function, and simplify the resulting expression to get equilibrium consumption in terms of exogenous variables and parameters:

$$C^*(a, b, I_0, G_0, T_0) = [a + b(I_0 + G_0 - T_0)]/(1-b)$$

Since we have <u>explicit</u> reduced form solutions for Y* and C*, we can investigate the comparative statics of the model by partially differentiating Y* and C* with respect to each of the variables a, b, $I_0$, $G_0$, $T_0$ in turn. For example, partially differentiating C* with respect to $T_0$ yields

$$\frac{\partial C^*}{\partial T_0} = -\frac{b}{(1-b)} < 0$$

This comparative static result tells us that equilibrium consumption is negatively correlated with the level of taxation.

Now suppose that we do not want to restrict ourselves to a *linear* consumption function. Instead, we shall let consumption be some *general* function of Y and $T_0$, written as follows:

$$C = C(Y, T_0)$$

(We can assume that this function is differentiable everywhere ie. smooth and continuous). In this case, we cannot solve for equilibrium income in the way we did above (ie. by substituting the equation for C into the equilibrium condition and then solving for Y*) because we do <u>not</u> have an explicit functional form for C. The most we can do is substitute the general function $C = C(Y, T_0)$ into the equilibrium condition to get the following expression:

$$Y = C(Y, T_0) + I_0 + G_0$$

Assuming that an equilibrium point (Y*, C*) exists, and provided that certain other conditions are satisfied (to be discussed in a later lecture), we can assume that Y* and C* are given by reduced forms

$$Y^* = Y^*(I_0, G_0, T_0)$$
$$C^* = C^*(I_0, G_0, T_0)$$

as before, but now we do <u>not</u> have <u>explicit</u> forms for these functions. Thus, we cannot obtain an explicit expression for the comparative static partial derivative $\partial C^*/\partial T_0$, say, from a reduced form solution for C* in this general-function case.

We know that at equilibrium, the following identities hold:

$$Y^* \equiv C(Y^*, T_0) + I_0 + G_0$$
$$C^* \equiv C(Y^*, T_0)$$

***Note that these expressions are identities (they are called <u>equilibrium identities</u>) because they must always hold at any <u>equilibrium</u> point (Y*, C*), irrespective of what the actual values of Y* and C* are.*** Suppose we try to find the comparative static partial derivative $\partial C^*/\partial T_0$ from the equilibrium identity for C*. The rate of change of the function $C(Y^*, T_0)$ with respect to $T_0$ is <u>not</u> given by a partial derivative $\partial C(Y^*, T_0)/\partial T_0$ , because a change in $T_0$ will affect $C(Y^*, T_0)$ both <u>directly</u>, and also <u>indirectly</u> through Y*. In other words, the arguments of the function $C(Y^*, T_0)$ are not independent of each other, because a change in $T_0$ will affect Y*, so we cannot simply partially differentiate $C(Y^*, T_0)$ with respect to $T_0$. To find the rate of change of $C(Y^*, T_0)$ with respect to $T_0$ when Y* and $T_0$ are related, we must resort to <u>total</u> differentiation. Total differentiation is based on the concept of the <u>total differential</u> of a function, to which we now turn.

## 6.3. Total differentials

So far, we have used the symbol dy/dx to denote the derivative of a function $y = f(x)$ with respect to x. From now on, we shall think of dy/dx as a ratio of two quantities, dy and dx, called the differentials of y and x respectively.

Suppose that we are given a continuous and 'smooth' function $y = f(x)$. ***We define the differential of the independent variable x, denoted by dx, as an 'infinitesimally' small change in x***. (Note: an 'infinitesimally' small change in a variable is a change that is treated for mathematical purposes as approaching zero in the limit). Since x is related to y through the function $y = f(x)$, an infinitesimally small change in x will 'cause' a corresponding infinitesimally small change in y denoted by dy. We define this to be the differential of the dependent variable y. In other words, ***we define the differential of the dependent variable y, denoted by dy, as the 'infinitesimally' small change in y caused by the differential dx***. Now, we know that $dy/dx \equiv f'(x)$. If we interpret dy/dx as the ratio of dy and dx, we can rearrange this identity to get

$$dy \equiv f'(x)dx$$

This tells us that the ordinary derivative $f'(x)$ of the function $y = f(x)$ can be used to 'convert' a differential dx into the corresponding differential dy.

Example: Given the function $y = f(x) = 3x^2 + 7x - 5$ and a differential dx, find the differential dy.
Solution: The first step is to find $f'(x)$. Using elementary rules of differentiation for a function of a single variable, we have $f'(x) = 6x + 7$. Thus, the desired differential is $dy = (6x + 7)dx$. Chiang (Chapter 8, page 189) shows how this result can be used to approximate dy given a small increase in x from 5 to 5.01. ***Note that the identity $dy \equiv f'(x)dx$ holds only for 'infinitesimal' changes in x.*** For changes in x which are not infinitesimal, the actual change produced in y will differ from the value of dy calculated using $dy \equiv f'(x)dx$ by an amount which increases with the size of the change in x.

We now want to do something similar to this for a function of two variables, say $y = f(x_1, x_2)$. This equation represents a surface in 3-space, and the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ evaluated at a particular point are the slopes of the two lines which are tangential to the surface in the $x_1$ and $x_2$ directions at that point (see Chiang, Chapter 7, pages 176-177). In this case, we define the differentials of the independent variables, denoted by $dx_1$ and $dx_2$, as infinitesimally small changes in $x_1$ and $x_2$. We define the differential of the dependent variable y, denoted by dy, by the following equation:

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2$$

***The differential dy is called the total differential of y.*** As in the one-variable case, the above equation holds only for infinitesimal changes in $x_1$ and $x_2$, although it will provide a good approximation to the 'true' change in y when small changes in $x_1$ and $x_2$ are considered.

Example: Given the function $y = f(x_1, x_2) = (2x_1x_2)/(x_1 + x_2)$ and the differentials $dx_1$ and $dx_2$, find the total differential dy.
Solution: The first step is to find the two partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$. Using the quotient rule, $\partial y/\partial x_1 = [(x_1 + x_2)2x_2 - 2x_1x_2]/(x_1 + x_2)^2 = (2x_2^2)/(x_1 + x_2)^2$, and $\partial y/\partial x_2 = [(x_1 + x_2)2x_1 - 2x_1x_2]/(x_1 + x_2)^2 = (2x_1^2)/(x_1 + x_2)^2$. Thus, the required total differential is

$$dy = (2x_2^2)/(x_1 + x_2)^2dx_1 + (2x_1^2)/(x_1 + x_2)^2dx_2$$

***Note that the process of finding the total differential dy of a function is called <u>total differentiation</u>.***

Everything we have said about functions of two variables works just as well for functions of any number of variables. Thus, if $y = f(x_1, x_2, \ldots, x_n)$, then by definition the total differential dy is

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \cdots + \frac{\partial y}{\partial x_n}dx_n$$

and this equation provides a good approximation to the 'true' change in y when small changes in $x_1, x_2, \ldots, x_n$ are considered.

**6.4. <u>Total derivatives</u>**

Now suppose that we are given a function from $E^2$ to $E^1$, $y = f(x_1, x_2)$, in which $x_1$ and $x_2$ are themselves functions of another variable, say time t. In this case, y is really a function of only one underlying variable (time), ie. $y = y(t)$, so the way in which y varies with t is described by the ordinary derivative dy/dt for a function of one variable. If we were given full specifications of the functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$, we could substitute these into the function $y = f(x_1, x_2)$ to get an expression for y involving only the variable t, and we could then use the rules of differentiation for a function of a single variable to find the derivative dy/dt. However, there is an alternative approach which would almost certainly be easier, and which is conceptually useful with regard to the comparative static analysis of general-function models.

We saw earlier that if $y = f(x_1, x_2)$, then

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 \qquad (1)$$

We also know that the derivative dy/dt can be treated as a ratio of two differentials, dy and dt. Dividing (1) by the differential dt gives the <u>total derivative</u> of y with respect to t:

$$\frac{dy}{dt} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dt} \qquad (2)$$

We have thus divided the operation of finding the derivative dy/dt into a number of simpler parts. First, we find the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$. Then, we find the ordinary derivatives $dx_1/dt$ and $dx_2/dt$. Finally, we substitute these into formula (2) to get dy/dt.

We have also established an important principle. Suppose that, in the above example, $x_1 = t$. Thus, $y = f(x_1, x_2(x_1))$. Then (2) becomes

$$\frac{dy}{dx_1} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dx_1} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dx_1} = \underbrace{\frac{\partial y}{\partial x_1}}_{\text{direct effect}} + \underbrace{\frac{\partial y}{\partial x_2}\frac{dx_2}{dx_1}}_{\text{indirect effect}} \qquad (3)$$

This clearly illustrates the difference between the <u>partial</u> derivative $\partial y/\partial x_1$ , and the <u>total</u> derivative $dy/dx_1$. ***Unlike the partial derivative, the total derivative allows for any <u>indirect</u> effects on y of a change in $x_1$ , via the other variables in the function.***

In general, for a function of n exogenous variables $y = f(x_1, x_2, \ldots, x_n)$, the total derivative with respect to one of its variables, say $x_1$, is given by

$$\frac{dy}{dx_1} = \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dx_1} + \cdots + \frac{\partial y}{\partial x_n}\frac{dx_n}{dx_1}$$

Example: Given the function $y = f(x, w) = 3x - w^2$, where $x = g(w) = 2w^2 + w + 4$, find the total derivative dy/dw.

Solution: The crude way of doing this would be to substitute $x = g(w)$ into $y = f(x, w)$, to get $y = 3(2w^2 + w + 4) - w^2 = 5w^2 + 3w + 12$. Then using the rules of differentiation for a function of a single variable, $dy/dw = 10w + 3$. We can get the same result using formula (3) above:
$dy/dw = f_w + f_x(dx/dw) = -2w + 3(4w + 1) = 10w + 3$.

### 6.5. An application of the total differential approach to a simple IS-LM model

Suppose that the goods market is characterised by the following three equations:

$$Y = C + I + G$$
$$C = C(Y) \qquad (0 < C_Y < 1)$$
$$I = I(Y, r) \qquad (I_r < 0)$$

where Y, C, I and G denote the usual variables, and r denotes the interest rate. The third equation says that investment spending is a decreasing function of the interest rate. G is assumed exogenous.

Furthermore, suppose that the money market is characterised by the following two equations:

$$M_d = L(Y, r) \qquad (L_Y > 0,\ L_r < 0)$$
$$M_d = M_s$$

where $M_d$ and $M_s$ denote aggregate demand and supply of money respectively. The equation for $M_d$ says that money demand is an increasing function of income, and a decreasing function of the interest rate. $M_s$ is assumed to be exogenously given.

The principal aim in this section is to use the above model to make comparative static predictions about how the equilibrium values of income, Y*, and the interest rate, r*, will alter as we vary the exogenous variables G and $M_s$.

Simplifying the model, we find the following:

IS curve: $$Y - C(Y) - I(Y, r) = G \qquad (4)$$
LM curve: $$L(Y, r) = M_s \qquad (5)$$

***First, let us consider the slopes of the IS and LM curves respectively.***

Totally differentiating (4), we find

$$dY - C_Y dY - I_Y dY - I_r dr = dG$$

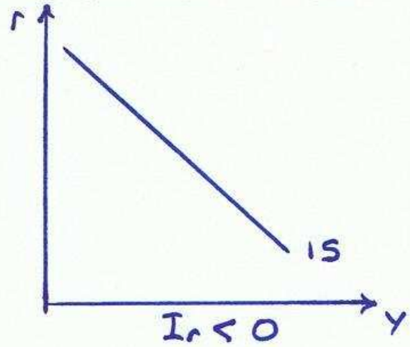
Collecting terms, this equation can be written as

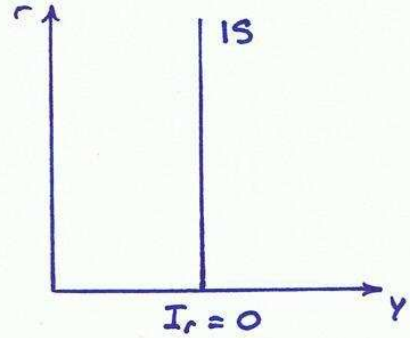
$$dY(1 - C_Y - I_Y) - I_r dr = dG \qquad (6)$$

For a given level of G (so that $dG = 0$), we can find dr/dY (ie. the slope of the IS curve): rearranging (6) gives $dY(1 - C_Y - I_Y) = I_r dr$. Thus

$$\left.\frac{dr}{dY}\right|_{G=G_0} = \frac{(1 - C_Y - I_Y)}{I_r}$$

Depending on the assumptions made about the partial derivatives, we can now draw the IS curve with the appropriate slope in (Y, r) space:




Let us follow the same procedure in order to examine the slope of the LM curve. First, we totally differentiate (5) to get

$$L_Y dY + L_r dr = dM_s \tag{7}$$

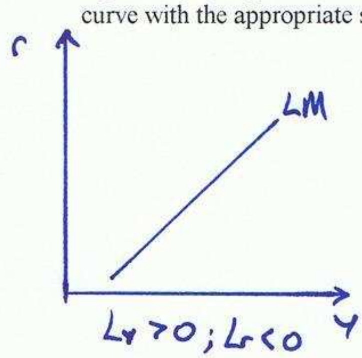
Assuming a <u>given</u> money supply, we must have $dM_s = 0$, so we get

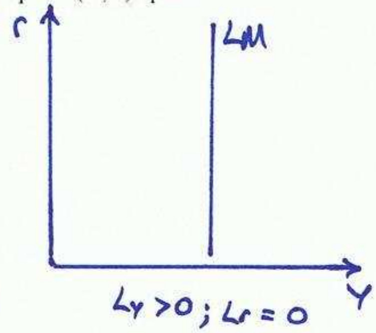
$$L_Y dY + L_r dr = 0$$

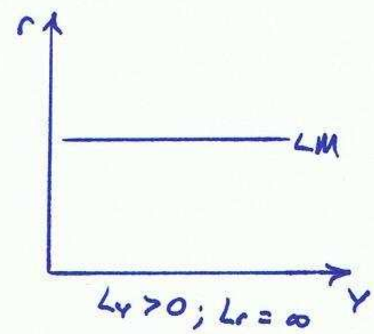
This yields

$$\left.\frac{dr}{dY}\right|_{M_s = \overline{M_s}} = -\frac{L_Y}{L_r}$$

Again, depending on the assumptions made about the partial derivatives, we can now draw the LM curve with the appropriate slope in (Y, r) space:




***Secondly, let us investigate the comparative statics of the IS-LM model.***

We can use matrix algebra on equations (6) and (7) (which have been linearised by total differentiation):

$$dY(1 - C_Y - I_Y) - I_r dr = dG \tag{6}$$
$$L_Y dY + L_r dr = dM_s \tag{7}$$

Putting this two-equation system in matrix form we get

$$\begin{bmatrix} (1-C_Y-I_Y) & -I_r \\ L_Y & L_r \end{bmatrix} \begin{bmatrix} dY \\ dr \end{bmatrix} = \begin{bmatrix} dG \\ dM_s \end{bmatrix} \qquad (8)$$

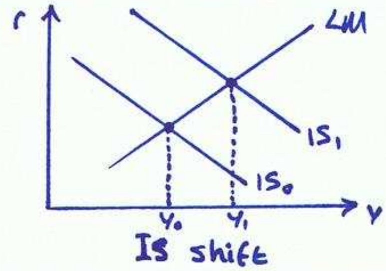
A unique solution to the system exists if and only if the coefficient matrix is invertible. Its determinant is $(1-C_Y-I_Y)L_r + I_rL_Y$, and we need this to be nonzero. Assuming (on the basis of economic theory) that $L_r < 0$, $L_Y > 0$, $I_r < 0$, $(1-C_Y-I_Y) > 0$, then $(1-C_Y-I_Y)L_r + I_rL_Y < 0$. Since the determinant is nonzero given the assumptions of the model, we can now solve (8) using either Cramer's rule, or by inverting the coefficient matrix.

Using Cramer's rule to solve (8) for dY, we find that

$$dY = \frac{\begin{vmatrix} dG & -I_r \\ dM_s & L_r \end{vmatrix}}{(1-C_Y-I_Y)L_r + I_rL_Y} = \frac{L_r dG + I_r dM_s}{(1-C_Y-I_Y)L_r + I_rL_Y}$$

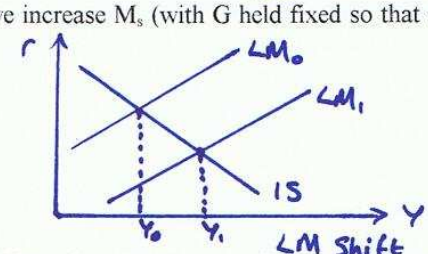
Hence, to find out how equilibrium Y alters as we increase G (with $M_s$ held fixed so that $dM_s = 0$), we divide both sides of this expression by dG:

$$\left.\frac{dY}{dG}\right|_{M_s=\overline{M_s}} = \frac{L_r}{(1-C_Y-I_Y)L_r + I_rL_Y} > 0$$




Similarly, to find out how equilibrium Y alters as we increase $M_s$ (with G held fixed so that dG = 0), we divide both sides of this expression by $dM_s$:

$$\left.\frac{dY}{dM_s}\right|_{G=G_0} = \frac{I_r}{(1-C_Y-I_Y)L_r + I_rL_Y} > 0$$




As part of your assignment for this week, you will be asked to complete the comparative static analysis of this model by finding $\left.\frac{dr}{dG}\right|_{M_s=\overline{M_s}}$ and $\left.\frac{dr}{dM_s}\right|_{G=G_0}$.

### 6.6. What you must do before the lecture next week (Thursday, 4th December, 'Implicit functions')

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter 8, pages 187-203, and make sure you are familiar with all the terms and concepts in these lecture notes. Do all the problems on the attached assignment sheet for Lecture 6. You will be asked to hand in your solutions at the start of the class next week.

There will not be a test in the class next week. Instead, I will go carefully over the material in this lecture, and over the solutions to the assignment for this week. In addition, I will give you a short lecture on *Jacobian determinants*, which you must have got to grips with by the Thursday lecture.

**(End of Lecture 6)**