Lecture 7. Implicit functions

7.1. Introduction: equations, functions and *implicit* functions

In the class test on Lecture 1 (sets and functions), I asked you to determine which of a set of graphs were graphs of functions, and which were not. We saw that, for a graph to be the graph of a function, there must be one, and only one, value of y for each value of x. Thus, a circular graph (for example) is not the graph of a function, because there are some values of x for which there are two corresponding values of y. However, a non-vertical linear graph is the graph of a function, because there can only be one value of y for each value of x.

Irrespective of whether or not a graph is the graph of a function, we may be able to interpret it as representing an equation relating x and y. For example, the graphs I gave you might represent the following equations:

(a). $y = -2x + 7$ (b). $y = -x^2 + 6x - 8$ (c). $x = y^2$

(d). $y = x^2$ (e). $(x-10)^2 + (y-10)^2 = 64$ (f). $x = 4$

This illustrates a very important point: ***all functional relationships can be written as equations, but not all equations are functions***. For an equation to be a function, it must satisfy the condition that for each value of the independent variable x, there is one and only one value of the dependent variable y.

The distinction between an equation and a function is important when we consider the concept of an *implicit* function. In the univariate context, ***an implicit function is a function of the form $y = f(x)$ which is implied by an equation of the form $F(y, x) = 0$***. For example, the equation $y - 3x^4 = 0$ is of the form $F(y, x) = 0$ and implies the function $y = 3x^4$ (which is of the form $y = f(x)$). Therefore $y = 3x^4$ is an implicit function of the equation $y - 3x^4 = 0$. Similarly, in the multivariate context, ***an implicit function is a multivariate function of the form $y = f(x_1, . . ., x_n)$ which is implied by an equation of the form $F(y, x_1, . . ., x_n) = 0$***.

Now, an explicit function $y = f(x_1, x_2, . . ., x_n)$ can always be written as $y - f(x_1, x_2, . . ., x_n) = 0$, which is an equation of the form $F(y, x_1, . . ., x_n) = 0$. But given an equation $F(y, x_1, . . ., x_n) = 0$, it is not always convenient, or even possible, to express y as a function of $x_1, x_2, . . ., x_n$ explicitly. For example, it can be shown that $F(y, x) = 2x^2 + 4xy - y^4 + 67 = 0$ implies a function of the form $y = f(x)$ over some subset of E^1, but the equation cannot easily be solved for y explicitly. As we saw above, we might also have an equation involving x and y (eg. $x^2 + y^2 - 64 = 0$) which is not a function at all (either implicit or explicit). Thus, we want to know the conditions under which an equation $F(y, x_1, . . ., x_n) = 0$ implicitly defines a function $y = f(x_1, . . ., x_n)$. These conditions are provided by the famous implicit function theorem, which you will often come across in your reading.

When setting up mathematical models in economics, we are often faced with situations in which we have equations which cannot be solved to get endogenous variables as explicit functions of exogenous variables and parameters. In the general-function national income model of the previous lecture, for example, we had the equations

$$Y = C(Y, T_0) + I_0 + G_0$$
$$C = C(Y, T_0)$$

Assuming that an equilibrium point (Y*, C*) exists, the implicit function theorem provides the conditions under which we can assume that implicit (reduced form) functions

$$Y^* = Y^*(I_0, G_0, T_0)$$
$$C^* = C^*(I_0, G_0, T_0)$$

exist. We can think about these functions (and their partial derivatives), despite the fact that we cannot obtain expressions for them explicitly, provided that the conditions of the implicit function theorem are satisfied.

7.2. The implicit function theorem

Suppose our model contains an equation $F(y, x_1, \ldots, x_n) = 0$ which we know is satisfied at a particular point $(y^*, x_1^*, \ldots, x_n^*)$ in E^{n+1}. In other words, we know that $F(y^*, x_1^*, \ldots, x_n^*) = 0$. The implicit function theorem tells us that a function $y = f(x_1, \ldots, x_n)$ is defined over some subset of E^n which contains the point $(x_1^*, \ldots, x_n^*)$, provided that the following two conditions are satisfied:

(i). $F(y, x_1, \ldots, x_n)$ has continuous partial derivatives $\partial F/\partial y, \partial F/\partial x_1, \partial F/\partial x_2, \ldots, \partial F/\partial x_n$.

(ii). At the point $(y^*, x_1^*, \ldots, x_n^*)$, we have $\partial F/\partial y \neq 0$.

If these conditions are satisfied, the implicit function theorem means that we can think about a function $y = f(x_1, \ldots, x_n)$ and its partial derivatives $\partial y/\partial x_1, \partial y/\partial x_2, \ldots, \partial y/\partial x_n$, even if our model only contains an equation $F(y, x_1, \ldots, x_n) = 0$ which cannot be solved for y. The partial derivatives $\partial y/\partial x_1, \partial y/\partial x_2, \ldots, \partial y/\partial x_n$ can be obtained directly from the equation $F(y, x_1, \ldots, x_n) = 0$ using a simple formula which is derived as follows. First, totally differentiate $F(y, x_1, \ldots, x_n) = 0$ to get

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x_1} dx_1 + \cdots + \frac{\partial F}{\partial x_n} dx_n = 0 \qquad (1)$$

Suppose we want to obtain an explicit expression for the partial derivative $\partial y/\partial x_i$. Recall that in order to partially differentiate $y = f(x_1, \ldots, x_n)$ with respect to x_i, we treat all the arguments of $f(x_1, \ldots, x_n)$ apart from x_i as constants. Thus, all the differentials of the exogenous variables apart from dx_i must be equal to zero, so equation (1) above reduces to

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x_i} dx_i = 0 \qquad (2)$$

Solving this equation for dy/dx_i we get

$$\left.\frac{dy}{dx_i}\right|_{\substack{\text{all other}\\ \text{variables}\\ \text{constant}}} \equiv \frac{\partial y}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial y} \qquad (3)$$

We can use this formula to get all the partial derivatives $\partial y/\partial x_1, \partial y/\partial x_2, \ldots, \partial y/\partial x_n$ directly from the equation $F(y, x_1, \ldots, x_n) = 0$, without first having to solve it for y.

Example: Find explicit expressions for the partial derivatives $\partial y/\partial x$ and $\partial y/\partial w$ of any implicit functions that may be defined by $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$ around the point $(y, x, w) = (1, 1, 1)$.

Solution: Clearly, it is not easy to solve the given equation for y. However, we have $F(1, 1, 1) = 0$ (verify this for yourself), so we know that the equation $F(y, x, w) = 0$ holds at the point (1, 1, 1). The partial derivatives of $F(y, x, w)$ are

$$\frac{\partial F}{\partial y} = 3y^2x^2 + xw \qquad \frac{\partial F}{\partial x} = 2y^3x + yw \qquad \frac{\partial F}{\partial w} = 3w^2 + yx$$

and these are all continuous. Thus, the first condition of the implicit function theorem is satisfied. If we evaluate the partial derivative $\partial F/\partial y$ at the point (y, x, w) = (1, 1, 1) we get $\partial F/\partial y = 3(1)^2(1)^2 + (1)(1) = 4 \neq 0$, so the second condition of the implicit function theorem is satisfied. It follows that an implicit function y = f(x, w) exists around the point (y, x, w) = (1, 1, 1). Using formula (3) above, we get the two desired expressions for the partial derivatives of y = f(x, w):

$$\frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{(2y^3x + yw)}{(3y^2x^2 + xw)} \qquad \frac{\partial y}{\partial w} = -\frac{\partial F/\partial w}{\partial F/\partial y} = -\frac{(3w^2 + yx)}{(3y^2x^2 + xw)}$$

7.3. <u>Extension of the implicit function theorem to the simultaneous equation case</u>

Often, our mathematical model will be in the form of a simultaneous equation system

$$\begin{aligned} F^1(y_1, \ldots, y_n, x_1, \ldots, x_m) &= 0 \\ F^2(y_1, \ldots, y_n, x_1, \ldots, x_m) &= 0 \\ \vdots \qquad\qquad & \ \vdots\ \vdots \\ F^n(y_1, \ldots, y_n, x_1, \ldots, x_m) &= 0 \end{aligned} \qquad (4)$$

which we <u>know</u> is satisfied at a particular point $(y_1^*, \ldots, y_n^*, x_1^*, \ldots, x_m^*)$. In other words, we <u>know</u> that

$$\begin{aligned} F^1(y_1^*, \ldots, y_n^*, x_1^*, \ldots, x_m^*) &= 0 \\ F^2(y_1^*, \ldots, y_n^*, x_1^*, \ldots, x_m^*) &= 0 \\ \vdots \qquad\qquad & \ \vdots\ \vdots \\ F^n(y_1^*, \ldots, y_n^*, x_1^*, \ldots, x_m^*) &= 0 \end{aligned}$$

In this case, the implicit function theorem tells us that a set of functions

$$\begin{aligned} y_1 &= f^1(x_1, \ldots, x_m) \\ y_2 &= f^2(x_1, \ldots, x_m) \\ \vdots \ & \quad \vdots \\ y_n &= f^n(x_1, \ldots, x_m) \end{aligned} \qquad (5)$$

is defined over some subset of E^m which contains the point $(x_1^*, \ldots, x_m^*)$, provided that the following two conditions are satisfied:

(i). $F^1, F^2, \ldots, F^n$ all have continuous partial derivatives with respect to all the variables $y_1, \ldots, y_n, x_1, \ldots, x_m$.

(ii). At the point $(y_1^*, \ldots, y_n^*, x_1^*, \ldots, x_m^*)$, the following Jacobian determinant is non-zero:

$$|J| = \begin{vmatrix} \partial F^1/\partial y_1 & \partial F^1/\partial y_2 & \cdots & \partial F^1/\partial y_n \\ \partial F^2/\partial y_1 & \partial F^2/\partial y_2 & \cdots & \partial F^2/\partial y_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial F^n/\partial y_1 & \partial F^n/\partial y_2 & \cdots & \partial F^n/\partial y_n \end{vmatrix} \neq 0$$

If these conditions are satisfied, we can find the partial derivatives of the implicit functions in (5) directly from the n equations in (4) above, without having to solve them for $y_1, \ldots, y_n$. The formula for doing this is derived as follows. First, we take the total differential of each equation in (4) above. This gives us a <u>system</u> of n differential equations of the form

$$\frac{\partial F^j}{\partial y_1}dy_1 + \cdots + \frac{\partial F^j}{\partial y_n}dy_n + \frac{\partial F^j}{\partial x_1}dx_1 + \cdots + \frac{\partial F^j}{\partial x_m}dx_m = 0 \quad (j = 1, \ldots, n) \qquad (6)$$

Suppose we are interested in the partial derivatives $\partial y_1/\partial x_i, \partial y_2/\partial x_i, \ldots, \partial y_n/\partial x_i$. Holding all exogenous variables constant apart from x_i, it must be the case that all the differentials of the exogenous variables apart from dx_i are zero. Thus, the system (6) above reduces to

$$\frac{\partial F^j}{\partial y_1}dy_1 + \cdots + \frac{\partial F^j}{\partial y_n}dy_n + \frac{\partial F^j}{\partial x_i}dx_i = 0 \quad (j = 1, \ldots, n) \qquad (7)$$

Dividing every equation in (7) throughout by dx_i, and taking the $\partial F^j/\partial x_i$ term in each equation to the right hand side gives us the system

$$\frac{\partial F^j}{\partial y_1}\frac{dy_1}{dx_i} + \cdots + \frac{\partial F^j}{\partial y_n}\frac{dy_n}{dx_i} = -\frac{\partial F^j}{\partial x_i} \qquad (j = 1, \ldots, n)$$

which can also be written in terms of *partial* derivatives as

$$\frac{\partial F^j}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \cdots + \frac{\partial F^j}{\partial y_n}\frac{\partial y_n}{\partial x_i} = -\frac{\partial F^j}{\partial x_i} \qquad (j = 1, \ldots, n) \tag{8}$$

since we are holding all exogenous variables apart from x_i constant. The final step is to realise that we can rewrite (8) in matrix form as

$$\begin{bmatrix} \partial F^1/\partial y_1 & \partial F^1/\partial y_2 & \cdots & \partial F^1/\partial y_n \\ \partial F^2/\partial y_1 & \partial F^2/\partial y_2 & \cdots & \partial F^2/\partial y_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial F^n/\partial y_1 & \partial F^n/\partial y_2 & \cdots & \partial F^n/\partial y_n \end{bmatrix} \begin{bmatrix} \partial y_1/\partial x_i \\ \partial y_2/\partial x_i \\ \vdots \\ \partial y_n/\partial x_i \end{bmatrix} = \begin{bmatrix} -\partial F^1/\partial x_i \\ -\partial F^2/\partial x_i \\ \vdots \\ -\partial F^n/\partial x_i \end{bmatrix} \tag{9}$$

The coefficient matrix in this system has a determinant which is $|J| \neq 0$ by condition (ii) of the implicit function theorem, so there must be a unique solution vector. Using Cramer's rule, this solution can be expressed as

$$\frac{\partial y_j}{\partial x_i} = \frac{|J_i|}{|J|} \qquad (j = 1, \ldots, n)$$

where J_i is the matrix obtained by replacing the ith column of the coefficient matrix in (9) with the (constant) vector on the right hand side of (9). Using a similar procedure, all the partial derivatives of the implicit functions in (5) with respect to the other exogenous variables $x_1, x_2, \ldots, x_n$ can also be obtained.

7.4. <u>An application to the comparative static analysis of a simple market model</u>

Suppose there is a single-commodity market described by the following equations:

$$Q_d = D(P, Y) \qquad (D_P < 0,\ D_Y > 0)$$
$$Q_s = S(P) \qquad (S_P > 0)$$
$$Q_d = Q_s$$

It is assumed that D(P, Y) and S(P) have continuous partial derivatives. The symbol Y denotes *income*, which is assumed to be exogenously determined. Note that this model comprises functions in general (not specific) form, and we cannot therefore solve explicitly for equilibrium price and quantity (P* and Q*) as functions of Y.

Setting $Q_d = Q_s = Q^*$ (ie. imposing the equilibrium condition on the model), we can rewrite the first two equations in the above system in implicit form as

$$F^1(P^*, Q^*; Y) = D(P^*, Y) - Q^* = 0$$
$$F^2(P^*, Q^*; Y) = S(P^*) - Q^* = 0 \tag{10}$$

We <u>know</u> that these equations must hold simultaneously at the equilibrium point (P*, Q*, Y). The conditions of the implicit function theorem are satisfied, since the demand and supply functions are both assumed to have continuous partial derivatives, so all the partial derivatives $\partial F^1/\partial P = D_P$, $\partial F^1/\partial Q = -1$, $\partial F^1/\partial Y = D_Y$, $\partial F^2/\partial P = S_P$, $\partial F^2/\partial Q = -1$, $\partial F^2/\partial Y = 0$ are continuous. In addition, $|J| \neq 0$, regardless of where it is evaluated:

$$|J| = \begin{vmatrix} \partial F^1/\partial P & \partial F^1/\partial Q \\ \partial F^2/\partial P & \partial F^2/\partial Q \end{vmatrix} = \begin{vmatrix} \partial D/\partial P & -1 \\ \partial S/\partial P & -1 \end{vmatrix} = \frac{\partial S}{\partial P} - \frac{\partial D}{\partial P} > 0$$

Thus, it must be true that $|J| \neq 0$ at the equilibrium point (P*, Q*, Y).

The implicit function theorem allows us to write the implicit functions $P^* = P^*(Y)$ and $Q^* = Q^*(Y)$, which have continuous derivatives $\partial P^*/\partial Y$ and $\partial Q^*/\partial Y$. These derivatives can be obtained directly from system (10) above.

Totally differentiate the two equations in (10) to get

$$\frac{\partial D}{\partial P^*} dP^* + \frac{\partial D}{\partial Y} dY - dQ^* = 0$$

$$\frac{\partial S}{\partial P^*} dP^* - dQ^* = 0$$

Dividing through by dY, and putting $\partial D/\partial Y$ (and $\partial S/\partial Y = 0$) on the right hand side we get

$$\frac{\partial D}{\partial P^*} \frac{dP^*}{dY} - \frac{dQ^*}{dY} = -\frac{\partial D}{\partial Y}$$

$$\frac{\partial S}{\partial P^*} \frac{dP^*}{dY} - \frac{dQ^*}{dY} = 0$$

This sytem can be written in matrix form as

$$\begin{bmatrix} \partial D/\partial P^* & -1 \\ \partial S/\partial P^* & -1 \end{bmatrix} \begin{bmatrix} dP^*/dY \\ dQ^*/dY \end{bmatrix} = \begin{bmatrix} -\partial D/\partial Y \\ 0 \end{bmatrix}$$

Using Cramer's rule, you should verify for yourself that this system implies $dP^*/dY = D_Y/(S_P - D_P) > 0$, and $dQ^*/dY = S_P D_Y/(S_P - D_P) > 0$. Thus, both the equilibrium quantity and the equilibrium price are positively correlated with the level of income.

7.5. <u>What you must do before next week</u>

Read Chiang, *Fundamental Methods of Mathematical Economics*, Chapter 8, pages 204-227, and make sure you are familiar with all the terms and concepts in these lecture notes. Do all the questions in the attached assignment sheet for Lecture 7. You will get very similar questions in your exam on December 18th.

There will be a short written test at the start of the class next week, in which I will ask you to analyse the comparative statics of a simple modified version of the IS-LM model of the previous lecture (Lecture 6. Total differentials). Again, you will be required to do something very similar in your exam.

(End of Lecture 7)