Lecture 9. Constrained optimisation

9.1. Introduction

Problems of maximisation or minimisation subject to constraints are central to microeconomics, and also arise in macroeconomics and econometrics. The general problem can be stated as follows:

(CO) Choose $(x_1, \ldots, x_n)$ to maximise $y = f(x_1, \ldots, x_n)$ subject to $(x_1, \ldots, x_n) \in C$.

The set C is a subset of the domain of the function, within which all n-dimensional vectors of the form $(x_1, \ldots, x_n)$ satisfy one or more constraints. The function f is known as the objective function, and C is a subset of E^n known as the constraint set. Note that the case where the objective function is to be minimised (eg. a cost minimisation problem) is covered by this formulation, since ***minimisation of f is equivalent to maximisation of -f.***

The form the constraint set takes depends on the problem at hand. For example, if the problem is one of consumer utility maximisation, then the constraint set would be the set of quantities of the various goods that were non-negative and which satisfied the budget constraint.

As in the case of the unconstrained optimisation problem (U) in Lecture 8, a solution to (CO) may not exist. For example, consider the problem of maximising $f(x) = x$ subject to $x \in C$, where $C = \{x | x \geq 10\}$. There is no value of $x < \infty$ that solves this problem, as x (and therefore f) can be made arbitrarily larger without leaving the constraint set. ***There is a pair of conditions that are jointly sufficient for a solution to (CO) to exist. These are that (i) f is continuous; (ii) C is a compact (closed and bounded) set.*** However, these are not necessary for a problem to have a solution. Many economic problems, when set up in mathematical format, do not satisfy these conditions, but nevertheless have a solution. It is often quite easy to check existence once the particular problem is written down. Therefore from now on, we assume that (CO) has a solution.

Given the existence of a solution, the analysis of the problem (CO) proceeds by simplifying the description of the set C. In this course, you will be examined on the simple case, constrained optimisation with equality constraints. You will already have met examples of this in micro- and macroeconomics. You will not be examined on the more general case, constrained optimisation with inequality constraints, as this topic is usually reserved for postgraduate courses. However, I will give you some notes on this in due course (probably after Christmas), in case you encounter problems of this type in your research work next year. Unconstrained optimisation, constrained optimisation with equality constraints, and constrained optimisation with inequality constraints are covered in detail in Chapters 9, 11, 12 and 21 of Chiang, *Fundamental Methods of Mathematical Economics*. In my opinion, Chiang's exposition is a pedagogical 'masterpiece', and there is no better introductory treatment of this topic in any other textbook. I strongly recommend that you read him if you possibly can.

9.2. Constrained optimisation with equality constraints: first-order (necessary) conditions

Here, the idea is that the set of vectors $(x_1, \ldots, x_n)$ in C can be written as the set of vectors satisfying the implicit relationship $g(x_1, \ldots, x_n) = 0$, or more formally, $C = \{x_1, \ldots, x_n | g(x_1, \ldots, x_n) = 0\}$. Often, $g = 0$ is called the equality constraint, and g is known as the constraint function. So the problem (CO) becomes:

(CE) Choose $(x_1, \ldots, x_n)$ to maximise $y = f(x_1, \ldots, x_n)$ subject to $g(x_1, \ldots, x_n) = 0$.

Examples:
(i). Suppose the problem is to maximise $-(x_1^2 + x_2^2)$ subject to $x_1+x_2 = 1$. Then $f = -(x_1^2+x_2^2)$, and $g = 1- x_1 - x_2$, or $g = x_1 + x_2 - 1$.
(ii). Let x_1 and x_2 be a consumer's consumption levels of goods 1 and 2 respectively, and suppose that consumer preferences are represented by the utility function $u(x_1, x_2) = \beta_1\log_e x_1 + \beta_2\log_e x_2$. Also suppose that the budget constraint is $p_1x_1 + p_2x_2 = m$, where p_i is the price of good i, and m is income. Then the consumer allocation problem is to choose $x_1, . . ., x_n$ to maximise utility subject to the budget constraint. This is an equality constrained optimisation problem, with $f = \beta_1\log_e x_1 + \beta_2\log_e x_2$, and $g = m - p_1x_1 - p_2x_2$.

To solve an equality constrained optimisation problem, we begin by defining the Lagrangean function for the problem (CE) as

$$L(x_1, . . ., x_n, \lambda) = f(x_1, . . ., x_n) + \lambda g(x_1, . . ., x_n)$$

where λ is known as the Lagrange multiplier.

Examples:
In example (i) above, $L(x_1, x_2, \lambda) = -(x_1^2 + x_2^2) + \lambda(1 - x_1 - x_2)$. In example (ii) above,
$L(x_1, x_2, \lambda) = \beta_1\log_e x_1 + \beta_2\log_e x_2 + \lambda(m - p_1x_1 - p_2x_2)$.

The Lagrangean function gives us a neat way of writing down the necessary conditions for a solution to (CE).

***Theorem 1**: Given the Lagrangean function $L(x_1, . . ., x_n, \lambda) = f(x_1, . . ., x_n) + \lambda g(x_1, . . ., x_n)$ for problem (CE), any solution values $x_1^*, . . ., x_n^*$ for the problem must satisfy the following first-order conditions:*

$$\partial L/\partial x_i = \partial f/\partial x_i + \lambda \partial g/\partial x_i = 0 \qquad i = 1, . . ., n \qquad (1)$$

$$\partial L/\partial \lambda = 0 \qquad (2)$$

This result gives us a means of finding $x_1^*, . . ., x_n^*$, as well as the value of the Lagrange multiplier λ^* at the optimum, which has special significance (see below).

Examples:
(i). Consider example (i) above again. We had $L(x_1, x_2, \lambda) = -(x_1^2 + x_2^2) + \lambda(1 - x_1 - x_2)$, so conditions (1) and (2) of Theorem 1 become

$$\partial L/\partial x_1 = -2x_1 - \lambda = 0 \qquad (1)$$
$$\partial L/\partial x_2 = -2x_2 - \lambda = 0 \qquad (2)$$
$$\partial L/\partial \lambda = 1 - x_1 - x_2 = 0 \qquad (3)$$

This is a system of three equations in three unknowns (x_1, x_2, and λ). We can solve the system to get x_1^*, x_2^* and λ^* as follows. First, solve equations (1) and (2) for λ to get

$$-2x_1 = \lambda \quad \text{and} \quad -2x_2 = \lambda$$

Clearly, these equations imply that $x_1 = x_2$. Substituting this result into (3) gives

$$1 - 2x_2 = 0 \Rightarrow x_2^* = 1/2, \text{ and so } x_1^* = 1/2.$$

Then $\lambda^* = -2x_1^* = -2x_2^* = -1$.

(ii). Consider example (ii) above. We had $L(x_1, x_2, \lambda) = \beta_1\log_e x_1 + \beta_2\log_e x_2 + \lambda(m - p_1x_1 - p_2x_2)$, so conditions (1) and (2) of Theorem 1 become

$$\partial L/\partial x_1 = \beta_1/x_1 - \lambda p_1 = 0 \qquad (1)$$
$$\partial L/\partial x_2 = \beta_2/x_2 - \lambda p_2 = 0 \qquad (2)$$
$$\partial L/\partial \lambda = m - p_1x_1 - p_2x_2 = 0 \qquad (3)$$

Again, this is a system of three equations in three unknowns, which can be solved for x_1^*, x_2^* and λ^* as follows. First, rearrange (1) and (2) to get

$$\beta_1 = \lambda p_1x_1 \qquad \text{and} \qquad \beta_2 = \lambda p_2x_2$$

Adding these together gives us

$$\beta_1 + \beta_2 = \lambda(p_1x_1 + p_2x_2) = \lambda m \qquad (4)$$

where the second equality in (4) follows from condition (3). Solving for λ^* from (4) gives $\lambda^* = (\beta_1+\beta_2)/m$. Substituting this result into (1) gives

$$x_1^* = \frac{\beta_1}{\beta_1 + \beta_2}\frac{m}{p_1}$$

and substituting the result into (2) gives

$$x_2^* = \frac{\beta_2}{\beta_1 + \beta_2}\frac{m}{p_2}$$

9.3. <u>Second-order (sufficient) conditions</u>

As in the case of the unconstrained optimisation problem (U) in Lecture 8, first-order necessary conditions for a solution to (CE) above may not be sufficient. Sufficiency conditions are usually stated in the form of conditions on the sign definiteness of the Hessian of the Lagrangean function, evaluated at the point of interest. These conditions get very complicated when there are multiple constraints, and it is very difficult to apply them. In practice, mathematical economists often rely on considerations of concavity, or other considerations, to ensure that any stationary points they find are global maxima or minima. In what follows, we will focus on problems with two or three choice variables, and one or two constraints, as these are the ones you will most often come across. ***However, I will give you the general rule for problems with n choice variables and m constraints, and I expect you to know it for your exam (you should give it as part of your answer to any exam questions on this topic).*** Chiang (Chapter 12) goes into far greater detail, and you should refer to him for more complicated cases.

The second-order sufficient conditions for a problem with two choice variables and one constraint are as follows. (Note again that we only need to discuss the case of <u>maximisation</u>, since a minimisation problem can be converted to a maximisation problem simply by multiplying the objective function by -1). Suppose the problem is

$$\max_{\lambda, x_1, x_2} L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

By Theorem 1 above, the first-order conditions for a maximum are

$$\partial L/\partial x_1 = f_1 + \lambda g_1 = 0$$
$$\partial L/\partial x_2 = f_2 + \lambda g_2 = 0$$
$$\partial L/\partial \lambda = g(x_1, x_2) = 0$$

Differentiating each of these equations with respect to x_1, x_2 and λ, we get the second-order partials, which can be arranged in matrix form as follows:

$$\overline{H} = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial \lambda} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial \lambda} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & f_{11} + \lambda g_{11} & f_{12} + \lambda g_{12} \\ g_2 & f_{21} + \lambda g_{21} & f_{22} + \lambda g_{22} \end{bmatrix}$$

This matrix of second-order partial derivatives of the Lagrangean function is called the bordered Hessian. ***The second-order sufficient condition for a maximum in problems with two choice variables and one constraint is that the determinant of the above bordered Hessian, $|\overline{H}|$, be strictly positive.*** As an exercise, you should check the second-order conditions for problem (ii) studied above. We will work through problem (i) for practice.

Example (i) revisited:
Consider example (i) above again. We had $L(x_1, x_2, \lambda) = -(x_1^2 + x_2^2) + \lambda(1 - x_1 - x_2)$, so conditions (1) and (2) of Theorem 1 were

$$\partial L/\partial x_1 = -2x_1 - \lambda = 0 \qquad (1)$$
$$\partial L/\partial x_2 = -2x_2 - \lambda = 0 \qquad (2)$$
$$\partial L/\partial \lambda = 1 - x_1 - x_2 = 0 \qquad (3)$$

The critical values of x_1, x_2, and λ for the Lagrangean function are $x_1^* = x_2^* = 1/2$ and $\lambda^* = -1$. The bordered Hessian in this case is

$$\overline{H} = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial \lambda} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial \lambda} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

and $|\overline{H}| = [(0)(-2)(-2) + (-1)(0)(-1) + (-1)(0)(-1)] - [(-1)(-2)(-1) + (-1)(-1)(-2) + (0)(0)(0)] = 4 > 0$, so the second-order conditions for a maximum are satisfied.

Now consider a problem with three choice variables and one constraint. The problem can be expressed in terms of the Lagrangean function as follows:

$$\max_{\lambda, x_1, x_2, x_3} L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda g(x_1, x_2, x_3)$$

By Theorem 1 above, the first-order conditions for a maximum are

$$\partial L/\partial x_1 = f_1 + \lambda g_1 = 0$$
$$\partial L/\partial x_2 = f_2 + \lambda g_2 = 0$$
$$\partial L/\partial x_3 = f_3 + \lambda g_3 = 0$$
$$\partial L/\partial \lambda = g(x_1, x_2, x_3) = 0$$

Differentiating each of these equations with respect to x_1, x_2, x_3, and λ, putting the 16 resulting second-order partials into a matrix, we get the following bordered Hessian:

$$\overline{H} = \begin{bmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & f_{11} + \lambda g_{11} & f_{12} + \lambda g_{12} & f_{13} + \lambda g_{13} \\ g_2 & f_{21} + \lambda g_{21} & f_{22} + \lambda g_{22} & f_{23} + \lambda g_{23} \\ g_3 & f_{31} + \lambda g_{31} & f_{32} + \lambda g_{32} & f_{33} + \lambda g_{33} \end{bmatrix}$$

The second-order sufficient condition for a <u>maximum</u> in a problem with three choice variables and one constraint is that the <u>third</u> principal minor of $\overline{H}$ be strictly positive, and that the fourth principal minor (which is just the determinant $|\overline{H}|$) be strictly negative.

<u>Example</u>: Find the extreme point of

$$y = f(x_1, x_2, x_3) = x_1^5 x_2^{10} x_3^{15}$$

subject to the constraint that

$$x_1 + x_2 + x_3 = 6$$

and confirm that it is a maximum. (Hint: the determinant of the bordered Hessian of the Lagrangean function for this problem is negative).

<u>Solution</u>: The problem can be made easier to solve by taking the natural logarithm of the objective function. This does not affect the results in any way. The Lagrangean is

$$L(x_1, x_2, x_3, \lambda) = 5\log x_1 + 10\log x_2 + 15\log x_3 + \lambda(6 - x_1 - x_2 - x_3)$$

First-order conditions for a maximum are

$$\partial L/\partial x_1 = 5/x_1 - \lambda = 0 \qquad (1)$$

$$\partial L/\partial x_2 = 10/x_2 - \lambda = 0 \qquad (2)$$

$$\partial L/\partial x_3 = 15/x_3 - \lambda = 0 \qquad (3)$$

$$\partial L/\partial \lambda = 6 - x_1 - x_2 - x_3 = 0 \qquad (4)$$

From (1), we get that $x_1 = 5/\lambda$ (5)

From (2) we get that $x_2 = 10/\lambda$ (6)

From (3) we get that $x_3 = 15/\lambda$ (7)

Substituting these results into (4) we get

$$6 - 5/\lambda - 10/\lambda - 15/\lambda = 0 \text{ or}$$

$$6 - 30/\lambda = 0 \qquad \Rightarrow \lambda^* = 5$$

Substituting this result into (5), (6) and (7) gives $x_1^* = 1$, $x_2^* = 2$, $x_3^* = 3$. The value of the objective function at this point is $(1)^5(2)^{10}(3)^{15}$ = 14, 349, 931. The bordered Hessian is

$$\overline{H} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1/2 & 0 \\ -1 & 0 & 0 & -1/3 \end{bmatrix}$$

We are told that $|\overline{H}| < 0$, so we only have to confirm that the third principal minor is positive. The third principal minor is

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ -1 & 0 & -1/2 \end{vmatrix} = [0 + 0 + 0] - [-1 - 1/2] = 3/2 > 0$$

Since this is strictly positive, the extreme point is a maximum.

Finally, consider a general problem with three choice variables and two constraints. Letting g and h denote the two constraint functions, the problem can be expressed in terms of the Lagrangean function as follows:

$$\max_{\lambda_1,\lambda_2,x_1,x_2,x_3} L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda_1 g(x_1, x_2, x_3) + \lambda_2 h(x_1, x_2, x_3)$$

The first-order conditions for a maximum are

$$\partial L/\partial x_1 = f_1 + \lambda_1 g_1 + \lambda_2 h_1 = 0$$
$$\partial L/\partial x_2 = f_2 + \lambda_1 g_2 + \lambda_2 h_2 = 0$$
$$\partial L/\partial x_3 = f_3 + \lambda_1 g_3 + \lambda_2 h_3 = 0$$
$$\partial L/\partial \lambda_1 = g(x_1, x_2, x_3) = 0$$
$$\partial L/\partial \lambda_2 = h(x_1, x_2, x_3) = 0$$

Differentiating each of these equations with respect to x_1, x_2, x_3, λ_1 and λ_2, and putting the 25 resulting second-order partials into a matrix, we get the following bordered Hessian:

$$\overline{H} = \begin{bmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{bmatrix}$$

(For brevity, the notation $L_{ij} \equiv \partial^2 L/\partial x_i \partial x_j$ is used above). ***The second-order sufficient condition for a <u>maximum</u> in a problem with three choice variables and two constraints is that the determinant of*** $\overline{H}$ ***,*** $|\overline{H}|$***, be strictly negative.*** Note that the determinant of $\overline{H}$ in this case is the fifth principal minor of $\overline{H}$.

<u>GENERAL RULE: PROBLEMS WITH n CHOICE VARIABLES AND m CONSTRAINTS</u>

The general rule, when the maximisation problem involves n choice variables and m constraints (m < n), is that starting with the minor of order (2m+1), the minors must alternate in sign, starting with sign $(-1)^{m+1}$.

You should verify that, in each of the examples discussed above, the second-order sufficient conditions are in accordance with this general rule.

9.4. <u>Interpretation of Lagrange multipliers</u>

By using a famous result called 'the Envelope Theorem' (which we do not have time to study in this course), it can be shown that ***each*** λ_i^* ***(ie. each Lagrange multiplier at the optimum) measures the increase in the objective function f made possible by a small relaxation of the corresponding constraint function***. For example, in the case of the constrained utility maximisation problem faced by the consumer in microeconomic theory, λ^* measures the increase in utility made possible by giving the consumer an additional unit of money (ie. by relaxing the budget constraint slightly).

9.5. <u>What you must do before the second in-class exam next week (Thursday, 18th December)</u>

Please attempt the constrained optimisation problems on the attached assignment sheet for Lecture 9. Learn the procedure for solving problems of this specific type (ie. take the natural logarithm of the objective function, form the Lagrangean, use the first-order conditions to solve for x_1, x_2, x_3 in terms of λ, etc.). You will face a very similar problem in your exam. Since we will not have time to go over these problems in a seminar, handwritten solutions are attached.

(End of Lecture 9)

Assignment for Lecture 9. Constrained optimisation

Question 1

Find the extreme point of

$$y = f(x_1, x_2, x_3) = x_1^3 x_2^2 x_3$$

subject to the constraint that

$$x_1 + x_2 + x_3 = 2$$

and confirm that it is a maximum. (Hint: the determinant of the bordered Hessian of the Lagrangean function for this problem is negative).

Question 2

Find the extreme point of

$$y = f(x_1, x_2, x_3) = x_1^4 x_2^8 x_3^{12}$$

subject to the constraint that

$$x_1 + x_2 + x_3 = 6$$

and confirm that it is a maximum. (Hint: the determinant of the bordered Hessian of the Lagrangean function for this problem is negative).

Question 3

Find the extreme point of

$$y = f(x_1, x_2, x_3) = x_1^9 x_2^6 x_3^3$$

subject to the constraint that

$$x_1 + x_2 + x_3 = 6$$

and confirm that it is a maximum. (Hint: the determinant of the bordered Hessian of the Lagrangean function for this problem is negative).