Lecture 11. Introduction to econometric models

11.1. What is econometrics ?

Almost every econometrics textbook begins by trying to 'define' what econometrics is, but there is no generally accepted definition. To start with, it is probably best to think about econometrics simply as the field in which we study economic phenomena by applying statistical methods to observed data. Statistical techniques are used for three main purposes: (1). to estimate economic relationships; (2). to test hypotheses involving economic behaviour; (3). to forecast the behaviour of economic variables.

This section provides a brief overview of what econometrics is about, and introduces a lot of basic concepts that you will meet again and again as you continue your studies. They will become much clearer to you as the year proceeds. ***Do not spend too much time trying to understand everything in this section! For the time being, just treat it as 'background reading' to get a feeling for what econometrics is all about. From an exam point of view, the crucial material of this lecture begins in Section 11.3. At all costs, you must get fully to grips with this material before next week, or you will find the course difficult to follow from now on. Please read the handout carefully, and seek help immediately if you find yourself struggling.*** (But do not come and see me if you have not bothered to read the handout first. It is a waste of my time).

An econometrician typically begins by setting up a theoretical model i.e. a logical framework for thinking about economic behaviour. The model might arise out of formal economic theory (involving the optimisation of some objective function subject to constraints), other studies, past experience, intuition etc.. In general, the idea is that the model should encapsulate theoretical insights into how economic forces may give rise to certain patterns of behaviour. The theoretical model is then 'translated' into an empirical model (also called an econometric model) of the relationships between relevant variables, such as income and consumption, which can be estimated using statistical methods like regression analysis. A well-specified econometric model is one which provides a reasonable approximation to the stochastic process that generates the observed data. This stochastic process is generally referred to as the data generating process, or the DGP. The econometric model might be formulated as a single equation model, such as a firm's cost function or production function, or as a structural equation model (also called a simultaneous equation model), comprising a system of equations that characterises the interdependence among variables. An example of the latter is a complete macroeconomic model, which may be used by the Treasury for policy analysis.

A single-equation model generally has the form

$$Y = f(X_1, X_2, \ldots, X_k, u)$$

where Y is a variable of primary interest, referred to as the dependent variable, or the regressand, or the endogenous variable, the X_i's are variables that have causal effects on the dependent variable (and are variously referred to as the independent variables, or the regressors, or the exogenous variables), and u is an unobservable random variable called the disturbance term, or the error term, which is supposed to capture random fluctuations and uncertainties that cannot be pre-specified in the model. (The X_i's may also be random variables, but they are usually taken to be observable along with Y). As an example, Y could be the earnings of an employee in a firm, X_1 might be the employee's age, X_2 the number of years of education, X_3 the number of years of work experience, etc.. ***If we draw a sample of workers and measure the attributes described above, not all of them will have exactly the same relationship between Y and the X's.*** An estimated relation will instead be a 'statistical average'. To allow for this fact, an econometric model will be formulated with an additional variable (denoted by u in the above equation) that captures the uncertainty in the relationship. A particularly simple form of an econometric model is given by the following equation, in which all the variables appear linearly:

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k + u$$

α and the β's are unknown parameters to be estimated from the data.

The independent variables denoted by X_i in the above equation might also be past values of the dependent and independent variables. The econometric model is then called a dynamic model. To illustrate, suppose Y_t is the consumption expenditure of a family, evaluated at time t. Families typically maintain their past standard of living, but adjust it if their financial position changes. Let X_t be the family's income at time t. ***If income fell from time period t-1 to time period t, we would expect consumption expenditure to be adjusted downward to accommodate the reduction in income.*** The following econometric model captures the underlying behaviour specified here:

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2(X_t - X_{t-1}) + u_t$$

We would expect β_1 to be positive because of the assumption that consumers try to maintain their standard of living (a positive β_1 means that a higher Y_{t-1} leads to a higher Y_t; a negative β_1 would imply that a higher Y_{t-1} leads to a lower Y_t). β_2 is also likely to be positive, because an increase in income (i.e. $(X_t - X_{t-1}) > 0$) would probably induce additonal consumption. The random error term u_t is included to capture the uncertainty in the postulated behavioural equation.

A structural equation model is generally of the form

$$f_i(Y_1, Y_2, \ldots, Y_G, X_1, X_2, \ldots, X_K, u_1, u_2, \ldots, u_G) = 0 \qquad i = 1, 2, \ldots, G$$

where the Y's are the endogenous variables, X's are the exogenous variables, and the u's are unobserved error terms. To illustrate, consider the following macroeconometric model:

$$C_t = \alpha_0 + \alpha_1 C_{t-1} + \alpha_2 Y_t^d + u_{1t}$$
$$I_t = \beta_0 + \beta_1 r_t + \beta_2 r_{t-1} + \beta_3 Y_t^d + u_{2t}$$
$$r_t = \gamma_0 + \gamma_1 Y_t + \gamma_2 M_t + \gamma_3 M_{t-1} + u_{3t}$$
$$Y_t^d = Y_t - T_t$$
$$Y_t = C_t + I_t + G_t$$

Y is national income, C is aggregate consumption, I is aggregate investment, G is government expenditure, T is taxes, M is money supply, and Y^d is disposable income, all measured at the time period t. The other variable in the system is the interest rate r. The endogenous variables in the system are C, I, r, Y^d and Y, which are jointly determined (i.e. they appear in each other's equations; for example, Y_t appears in the equation for C_t through the variable Y_t^d, and at the same time, C_t appears in the equation for Y_t). T, G and M are given exogenously. The variables M_{t-1}, I_{t-1} and Y_{t-1}^d are known as lagged variables (in certain cases, predetermined variables) because at time t their values are known. The variables u_1, u_2, and u_3 represent unobservable error terms that capture the uncertainty in the relationships. The fourth equation in the model above is called an accounting identity. It is assumed always to hold 'by definition', so it does not have an error term. The last equation is an equilibrium condition equating intended aggregate demand to national income. These two equations do not contain any unknown parameters.

Based on data on all the observable variables, the econometrician would be interested in estimating the unknown parameters. In this example, these are the α's, β's, γ's, and the parameters that pertain to the distributions of the random variables u_1, u_2, and u_3. The econometrician would also like to submit the model to a variety of diagnostic tests to make sure that it is a reasonable representation of the DGP. Methods of hypothesis testing would be useful not only at this diagnostic testing stage, but also to test the validity of various implications of the theoretical model on which the econometric model is based. If the estimated relationships obtained are broadly consistent with what a particular economic theory suggests should be observed, this can be used to try to persuade others that the theory is a good way of conceptualising what is going on 'in real life'. If the estimated relationships are difficult or impossible to reconcile with the theory, it may be the case that some of the assumptions of the economic theory are wrong, and that a new theory is called for. Alternatively, it may just be that we have employed the wrong statistical methods given the properties of data we are using, or that we have made inappropriate assumptions in moving from the theoretical model to the empirical model. What is 'right' and 'wrong' in this context therefore depends on the theoretical economic model itself, on the empirical model we are using to test the theory, and on the characteristics of the data we are using. To a large extent, econometrics is about developing appropriate empirical models for testing theories, and about finding the 'best' way to estimate relationships between observed variables using statistical methods, given adequate knowledge of the strengths and weaknesses of the data set.

Unlike the natural sciences (e.g. physics) where a researcher can usually conduct a controlled laboratory experiment, economics most frequently deals with non-experimental data generated by an incredibly complicated process involving the interactions of the behaviour of numerous economic and political agents. This creates a great deal of uncertainty in the models and methods used by econometricians. In particular, estimated relationships are never precise, hypothesis testing can lead to the error of rejecting a true hypothesis or failing to reject a false hypothesis, and forecasts of variables often turn out to be far from the values that are eventually observed.

The empirical model that an econometrician formulates is meant to be a description of the DGP. The actual data, however, are treated as one of several possible realisations of events. The theory of probability is used to construct a framework that portrays the likelihood of one type of realisation or another. The probability framework invariably depends on a number of unknown parameters. In order to estimate the parameters, an analyst typically obtains a sample of observations and uses them in conjunction with a probability model. Statistical inference deals with the methods of obtaining these estimates, measuring their precision, and testing hypotheses on the parameters in question.

11.2. Time series data, cross-section data and panel data

Since econometrics is about analysing data, it is logical to begin our study of the subject by considering the different types of data sets economists and econometricians use. At this stage, you only really need to be aware of three types, all of which we will discuss in the theoretical lectures and/or in the applied computer work.

A time series data set is just an ordered set of observations over time of one or more quantities you are interested in. For example, you might have a data set consisting of annual observations of aggregate consumer's expenditure and income for a particular country over the period 1900 to 1996. Letting C and Y denote aggregate consumption and income respectively, the time series data set would consist of 97 observations of each of these variables, arranged in time order: C_{1900}, C_{1901}, ..., C_{1996} and Y_{1900}, Y_{1901}, ..., Y_{1996}. A great deal of work has been done on the development of econometric techniques for the analysis of time series data, and there are a number of world famous texts dealing exclusively with this topic.

A cross-section data set is just a set of observed quantities at one point in time for each member of a sample of individuals or households. So suppose we take a random sample of N individuals from some large population at a particular point in time, and obtain information from each individual about his/her income (denoted again by Y), age (denoted by A) and number of years of schooling (denoted by E). We always assign a unique number from the set {1, 2, ..., N} to each individual so as to be able to distinguish one individual from another, and speak in general terms of 'the ith individual, for i = 1, 2, ..., N'. Then the cross-section data set consists of three observations, Y_i, A_i, and E_i for each i = 1, 2, ..., N, a total of 3N observations in all.

Panel or longitudinal data sets are made up by collecting cross-section data on a given sample of individuals at each of two or more different time points. You thus get a 'time series' of cross-sectional observations. We shall analyse a real panel data set from the British Household Panel Survey (BHPS) in our computing work later this term. You might well end up using the BHPS data set for your dissertation next year!

11.3. A simple two-equation model of national income determination

We shall introduce some fundamental concepts by means of a simple two-equation Keynesian national income model (you might find it helpful to revise Supplementary Lecture 1 from the mathematical methods course last term). The first element of the model is a consumption function, describing the relation between consumption and income. Using conventional notation, we denote consumption by C and income by Y, so that the consumption function can be written in general terms as

$$C = C(Y)$$

Recall from the discussion in Supplementary Lecture 1 that Keynes argued two things: First, that consumption increases as income increases, but not by as much as the increase in income ie. $0 < C_Y < 1$, where $C_Y = \partial C/\partial Y$ is the marginal propensity to consume. Second, that the proportion of income consumed decreases as income increases ie. that $\partial(C/Y)/\partial Y < 0$. Any mathematical function with these two properties is consistent with Keynes' two arguments, and for some purposes in economic theory, it is not necessary to say anything more about the function C(Y) (eg. recall the work we did last term on the comparative static analysis of general function national income models using total differentiation). However, ***in order to use a consumption function as a framework for data analysis, it is necessary to specify an explicit functional form***. The simplest possibility, useful at least as a first approximation, is to assume that consumption is a *linear* function of income ie.

$$C = \alpha + \beta Y$$

The first derivative, C_Y, is equal to the slope coefficient β, and the proportion of income consumed decreases as income increases if the intercept term α is positive (to see this, note that $C/Y = \alpha/Y + \beta$, so $\partial(C/Y)/\partial Y = -\alpha/Y^2 < 0 \Rightarrow \alpha > 0$). Thus, the linear equation above is consistent with Keynes' arguments if the coefficients satisfy

$$\alpha > 0 \qquad 0 < \beta < 1$$

The consumption function describes consumers' behaviour, and is therefore an example of what is known as a behavioural relation. ***Relevant data might consist of time series observations on aggregate consumers' expenditure and income over time, or cross-section data relating to a sample of households at a particular point in time.*** We shall variously use the subscripts t or i to index particular observations on variables. C_t and Y_t will usually denote aggregate consumption and income at time t, while C_i and Y_i will usually denote the consumer expenditure and income of the ith household in a cross-section sample. In either case, the number of observations in the sample is denoted by N, so the range of the subscripts is indicated as t = 1, . . ., N or i = 1, . . ., N. ***We assume that the coefficients α and β are constant across the sample of N observations, and so refer to them as parameters.*** Parameters are usually denoted by Greek letters in econometric modelling.

The model we are considering at the moment is a national income determination system, and an aggregate time series scenario is envisaged. We complete the model by writing the equilibrium condition:

$$Y_t = C_t + I_t$$

This divides income into two components of demand, the second being aggregate investment spending by firms.

We now have a pair of equations that together describe the determination of the values of the variables C and Y in terms of the values of the variable I and parameters α and β. ***The variables determined within the system are called endogenous variables, while those determined outside the system (about which the system has nothing to say) are called exogenous variables.*** The statement of the relations among the variables, together with a classification of the variables as endogenous or exogenous, comprises the economic model (it is not yet an econometric model, because we have not specified fully how the data are being generated; we will do this shortly). Thus, the model here is

$$C_t = \alpha + \beta Y_t$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t
exogenous variable: I_t

Notice that this model is a structural (or 'simultaneous') equation system. What we would ideally like to do is apply an empirical version of this model to data, in order to get estimates for the parameters α and β. Unfortunately, however, structural equations like these cannot easily be estimated directly (we shall see why later in the course). ***We need to convert our structural equations into reduced form equations, because the parameters of reduced form equations can be estimated more readily.*** Once we have obtained estimates of the parameters of the reduced form equations, we can sometimes (but not always) convert these estimates 'back' into estimates of the parameters of the structural equations. Usually, it is the structural parameters (not the reduced form parameters) we are really interested in.

To get an idea of how all this works, let us solve for the endogenous variables in terms of parameters and exogenous variables alone i.e. let us obtain the reduced form of the Keynesian national income model (we did something very similar in Supplementary Lecture 1 last term). Let us consider this solution algebraically. Substituting the income equation into the consumption function gives

$$C_t = \alpha + \beta(C_t + I_t)$$

and on rearranging we obtain

$$C_t = \frac{\alpha}{1-\beta} + \frac{\beta}{1-\beta} I_t$$

Similarly, substituting the consumption function into the income equation gives

$$Y_t = \alpha + \beta Y_t + I_t$$

which upon rearranging gives

$$Y_t = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I_t$$

We can rewrite these equations using new symbols for the coefficients as follows:

$$C_t = \gamma_0 + \gamma_1 I_t$$
$$Y_t = \delta_0 + \delta_1 I_t$$

This last pair of equations is the reduced form of the model. Each reduced form equation expresses a single endogenous variable in terms of exogenous variables and parameters. In this simple example, there is only one exogenous variable (I_t). The parameters of the structural equations (α and β in this case) are simply referred to as structural parameters. The parameters of the reduced form equations (γ_0, γ_1, δ_0, and δ_1) are called reduced form parameters. It is obvious by inspection that the reduced form parameters are defined in terms of the parameters of the structural equation model as follows:

$$\gamma_0 = \frac{\alpha}{1-\beta} \qquad \gamma_1 = \frac{\beta}{1-\beta}$$

$$\delta_0 = \frac{\alpha}{1-\beta} \qquad \delta_1 = \frac{1}{1-\beta}$$

Notice that there are four reduced form parameters but only two structural parameters. We can easily solve these four equations to get the structural parameters expressed in terms of the reduced form parameters as follows: substituting δ_1 into the equation for γ_0 we get

$$\gamma_0 = \delta_1\,\alpha$$

which implies that

$$\alpha = \gamma_0/\delta_1$$

Rearranging the equation for δ_1 we get

$$1\text{-}\beta = 1/\delta_1$$

which implies that

$$\beta = 1 - 1/\delta_1 = (\delta_1\text{-}1)/\delta_1$$

Now suppose that using our time series data, we estimate the two reduced form equations of our model (we shall learn how to do this later) and obtain the following estimates for the reduced form parameters: $\gamma_0* = 40$, $\gamma_1* = 3$, $\delta_0* = 40$, $\delta_1* = 4$. What are the implied values of the parameters of the structural equation (i.e. α and β) ? Well, we can use the two expressions above for the structural parameters in terms of the reduced form parameters to get the estimates $\alpha* = \gamma_0*/\delta_1* = 40/4 = 10$, and $\beta* = (\delta_1*\text{-}1)/\delta_1* = (4\text{-}1)/4 = 3/4$. ***These estimates of the parameters of the structural equation model appear to be consistent with our economic theory, since $\alpha* > 0$ and $0 < \beta* < 1$.*** We can write the behavioural relation between C and Y that we were originally interested in as

$$C = 10 + (3/4)Y$$

We could use this estimated equation for a number of purposes.

Unfortunately, it is not always possible to deduce what the values of the parameters of the structural equations are, when you are given a set of estimated reduced form parameters. This is because it is not always possible to 'convert back' from reduced form parameters to structural parameters. Econometricians call the question of when we can and when we can't deduce the values of the structural parameters the identification problem. They ask themselves: 'is it possible to identify the structural parameters of this model?', or: 'are the structural parameters of the model identified ?'. What they mean is: ***'can we use estimates of the reduced form parameters to get estimated values for the structural parameters?'***. We shall devote the whole of the next lecture to finding a general solution for the identification problem, because it is so important.

11.4. A demand and supply example

We shall now use a second example to further develop and emphasise some basic ideas. We consider the demand and supply of a commodity that is subject to an excise tax. (An excise tax, also called a quantity tax, is a tax that must be paid by the consumer for every unit that is bought.

From the viewpoint of the consumer, the tax is just like a higher price). It is assumed that the price of the commodity adjusts so as to equate demand and supply in every period. The behaviour of buyers of the commodity is described by a demand function, which says that Q_d, the quantity demanded, is a function of the market price P of the commodity, with a negative slope:

$$Q_d = Q_d(P) \qquad \partial Q_d/\partial P < 0$$

Once more, for practical purposes, we take this function to be linear, and so write

$$Q_d = \alpha_0 + \alpha_1 P \qquad \alpha_1 < 0$$

The behaviour of sellers of the commodity is described by a supply function, in which Q_s, the quantity supplied, is an increasing function of the price received by sellers, namely P-E, the price before tax, where E is the per unit excise tax. Again assuming linearity we have:

$$Q_s = \beta_0 + \beta_1(P-E) \qquad \beta_1 > 0$$

The equilibrium condition is $Q_d = Q_s$. Thus, we have three equations describing the determination of the variables P, Q_d and Q_s in terms of parameters and the excise tax variable. Setting $Q_d = Q_s = Q$, we get a two-equation structural model:

$$Q = \alpha_0 + \alpha_1 P$$
$$Q = \beta_0 + \beta_1(P-E)$$

endogenous variables: Q, P

exogenous variable: E

Each of the structural equations is a behavioural relation, describing the behaviour of one particular group or sector in the economy. Solving the demand and supply equations simultaneously for Q and P gives the reduced form. On equating the right hand sides we have

$$\alpha_0 + \alpha_1 P = \beta_0 + \beta_1(P-E)$$

and on rearranging we have the reduced form equation for price:

$$P = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \frac{\beta_1}{\beta_1 - \alpha_1} E$$

Then substituting for P in either of the structural equations and simplifying gives

$$Q = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\beta_1 - \alpha_1} + \frac{\alpha_1\beta_1}{\beta_1 - \alpha_1} E$$

Again, each reduced form equation expresses a single endogenous variable in terms of exogenous variables (only one in our examples) and parameters. On rewriting the reduced form equations by using new symbols for their coefficients, we have

$$P = \gamma_0 + \gamma_1 E$$
$$Q = \delta_0 + \delta_1 E$$

It is obvious by inspection that the expressions for the reduced form coefficients in terms of the structural parameters are

$$\gamma_0 = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \qquad \gamma_1 = \frac{\beta_1}{\beta_1 - \alpha_1} \qquad \delta_0 = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\beta_1 - \alpha_1} \qquad \delta_1 = \frac{\alpha_1\beta_1}{\beta_1 - \alpha_1}$$

Notice that now there are four reduced form parameters and four structural parameters. As in the Keynesian national income example in the previous section, we can solve back from the reduced form coefficients to the structural parameters. The algebraic solution is:

$$\alpha_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1} \qquad \alpha_1 = \frac{\delta_1}{\gamma_1} \qquad \beta_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1 - 1} \qquad \beta_1 = \frac{\delta_1}{\gamma_1 - 1}$$

(You will be asked to derive these expressions as part of your assignment for this week). ***Thus, having obtained estimates of the reduced form coefficients from data, unique values of the structural parameters could be deduced, and we therefore say that the structural parameters are identified.*** As already noted, this is (unfortunately) not always possible.

11.5. The disturbance term

Let us focus on the equation $C_t = \alpha + \beta Y_t$ for the moment. We are attempting to capture the systematic or deterministic part of the empirical relationship we think exists between C_t and Y_t. We are trying our best to come up with a model that will 'explain' or 'predict' C_t in terms of Y_t and a couple of parameters (α and β). At the moment, however, our model implies that if we knew the true values of α and β, we would never need to 'observe' C_t again, because we would always be able to calculate it exactly (without ever being wrong) by plugging our observation of Y_t for each period t into the formula $\alpha + \beta Y_t$. Clearly, this is not how things work in the 'real world'. It is a bit like pretending that we could throw darts at a dart board and hit the bull's-eye every single time without ever missing. In real life, we must always make room for a bit of 'randomness' in the things we do. Even if we knew the true values of α and β (which we don't), there would almost always be a difference between the amount of consumption spending we observe in period t (ie. C_t), and the amount the linear model predicts (ie. $\alpha + \beta Y_t$), not least because human behaviour is always a little bit unpredictable!

To connect the systematic part of the postulated relationship between C_t and Y_t to the 'real world', we therefore need to add a non-systematic or stochastic part to the model ('stochastic' just means 'random' or 'probabilistic'). This can be assumed to reflect the combined effect on C_t of many small and unpredictable factors which cannot be identified or measured. We represent the stochastic part of the model by adding a disturbance term (denoted by u_t) to the linear equation connecting C_t and Y_t, giving us

$$C_t = \alpha + \beta Y_t + u_t$$

As is always done with things that are 'random' or 'probabilistic', it is assumed that the unobservable disturbance term u_t takes on particular values more often or less often depending on how 'probable' these particular values are. The probabilities of all the values the disturbance term can take on can be assumed to be described by a probability density function.

Although the disturbance term u_t can potentially take on any one of a whole range of values, we always assume that the mean or expected value of the disturbance term is zero:

$$E[u_t] = 0$$

This is an essential assumption because, as we shall see later, our techniques for estimation are designed to work properly only if we have done everything we can to ensure that this assumption is correct. ***Saying that the disturbance term has an expected value of zero is basically a way of formalising the notion that we have tried our best to come up with a model for 'explaining' or 'predicting' C_t for each t, so that we have no a priori reason to expect the disturbance terms to take on any values other than zero.*** In other words, $E[u_t] = 0$ says that after trying our best to specify the model correctly, our best way of predicting what u_t will turn out to be is to say that u_t will be zero. Yet another way of saying the same thing is that our best way of predicting C_t is to say that C_t equals $\alpha + \beta Y_t$ for some values of α and β.

It is just like throwing darts at a dart board, trying to hit the bull's-eye. We know that we're probably going to miss the bull's-eye, even if we aim for it really carefully, but we don't know whether the dart will end up above the bull's-eye, below it, or to the left or right etc. If, in a pub bet, we had to put money on one particular spot on the dart board where we thought the dart was going to land, which spot on the dart board would we put our money on ? We would put our money on the bull's-eye, of course, because that is what we're trying really hard to hit! Why would you particularly want to bet money on any other spot on the board, if you were not deliberately aiming for that spot? Similarly, if we try to predict the value of C_t by computing the sum $\alpha + \beta Y_t$, then as long as $E[u_t] = 0$, our prediction $\alpha + \beta Y_t$ is just as likely to be higher than the actual value C_t as it is to be lower than C_t. The best 'guess' we can make about our prediction is that it is going to be on target.

Often, we do not need to specify the probability distribution of the disturbance term, but we shall discuss later in the course certain situations in which such a specification is necessary. Then it is typically assumed that the disturbance term has a normal probability distribution with mean zero and variance denoted by σ_u^2, and this assumption is written

$$u \sim N(0, \sigma_u^2)$$

As before, we complete our model with an equilibrium condition, and having introduced the random error term we speak of the resulting equations as a stochastic model or an econometric model, which now is

$$C_t = \alpha + \beta Y_t + u_t$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variable: I_t

Note that we do not attach random disturbances to identities. In general, things like definitions, accounting identities, technical relations, equilibrium conditions etc.. hold exactly. For example, profit equals total revenue less total costs, the balance of trade equals exports minus imports, etc..

We now derive the reduced form of the stochastic model in exactly the same way as we did for the model with no disturbance term: substituting the equilibrium condition into the consumption function we get

$$C_t = \alpha + \beta(C_t + I_t) + u_t$$

Solving this for C_t we get

$$C_t = \frac{\alpha}{1-\beta} + \frac{\beta}{1-\beta} I_t + \frac{1}{1-\beta} u_t$$

Substituting the consumption function into the equilibrium condition we get

$$Y_t = \alpha + \beta Y_t + I_t + u_t$$

Solving this for Y_t we get

$$Y_t = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I_t + \frac{1}{1-\beta} u_t$$

So now we have two reduced form equations, each containing an error term. Defining $v_t \equiv u_t/(1-\beta)$, we can rewrite these equations as

$$C_t = \gamma_0 + \gamma_1 I_t + v_t$$
$$Y_t = \delta_0 + \delta_1 I_t + v_t$$

Note that we still have $E[v_t] = 0$, because by the rules for manipulating expected values discussed in Lecture 10 Part 1, we have $E[u_t/(1-\beta)] = E[u_t]/(1-\beta) = 0/(1-\beta) = 0$. We can still estimate the two reduced form equations. Suppose we did this, and obtained the following estimates for the reduced form parameters: $\gamma_0^* = 40$, $\gamma_1^* = 3$, $\delta_0^* = 40$, $\delta_1^* = 4$. We could still use these to derive estimates of the structural parameters $\alpha^* = 10$ and $\beta = 3/4$ in the same way as before, but now we must bear in mind that the prediction

$$C_t = 10 + (3/4)Y_t$$

is subject to error. Note, however, that the error has an expected value of zero.

The demand and supply example allows us to discuss a further aspect of stochastic models. Again, we add random disturbance terms to the behavioural equations, and write the stochastic model as

$$Q = \alpha_0 + \alpha_1 P + u_d$$
$$Q = \beta_0 + \beta_1(P-E) + u_s$$

endogenous variables: Q, P

exogenous variable: E

The random disturbance u_d is the non-systematic part of the demand function, representing the influence on buyers' behaviour of all factors other than price. Likewise, the error term in the supply function, u_s, captures the effect of omitted factors on sellers' behaviour. Each non-systematic part has an expected value of zero:

$$E[u_d] = E[u_s] = 0$$

The additional point I want to make in the context of this simple model is an extremely important one: ***in general, we would expect that some of the omitted factors that have a small accidental influence on buyers' behaviour also influence sellers in some way.*** For example, freak weather conditions in a particular year might affect *both* the demand for *and* the supply of an agricultural commodity. ***The possibility that some random factors influence both disturbance terms implies that the disturbances will be correlated, positively or negatively, and the methods of analysis and estimation that we use must allow for this possibility.*** We typically do not assume that the random disturbances are uncorrelated. As we saw in Lecture 10 Part 2, the general definition of the covariance of two random variables is the expected value of the product of their deviations from the mean; thus

$$Cov[u_d, u_s] = E[(u_d - E[u_d])(u_s - E[u_s])]$$

However, since the disturbances have zero means, this reduces to the expected value of their product:

$$Cov[u_d, u_s] = E[u_d u_s]$$

Likewise, the definition of the variance is simplified by the zero mean condition:

$$V[u_d] = E[u_d^2] \qquad V[u_s] = E[u_s^2]$$

The coefficient of correlation between the two variables is given by the covariance divided by the product of their standard deviations:

$$Corr[u_d, u_s] = Cov[u_d, u_s]/(\sqrt{V[u_d]}\sqrt{V[u_s]})$$

Recall that this quantity lies between -1 and 1, and in general is non-zero.

The reduced form of the stochastic demand and supply model is

$$P = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \frac{\beta_1}{\beta_1 - \alpha_1}E + \frac{u_d - u_s}{\beta_1 - \alpha_1}$$

$$Q = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\beta_1 - \alpha_1} + \frac{\alpha_1\beta_1}{\beta_1 - \alpha_1}E + \frac{\beta_1 u_d - \alpha_1 u_s}{\beta_1 - \alpha_1}$$

(You will be asked to derive these expressions as part of your assignment for this week). On rewriting these equations compactly by using new symbols, we get

$$P = \gamma_0 + \gamma_1 E + v_1$$
$$Q = \delta_0 + \delta_1 E + v_2$$

The reduced form disturbances, denoted v_1 and v_2, are related to the structural disturbances as follows:

$$v_1 = \frac{u_d - u_s}{\beta_1 - \alpha_1} \qquad v_2 = \frac{\beta_1 u_d - \alpha_1 u_s}{\beta_1 - \alpha_1}$$

Notice that each reduced form disturbance is a function of <u>both</u> structural disturbances. Since the structural disturbances have mean values of zero, so do the reduced form disturbances:

$$E[v_1] = \frac{E[u_d] - E[u_s]}{\beta_1 - \alpha_1} = 0 \qquad E[v_2] = \frac{\beta_1 E[u_d] - \alpha_1 E[u_s]}{\beta_1 - \alpha_1} = 0$$

The covariance between the reduced form disturbances is in general nonzero, irrespective of any covariance between the structural disturbances, because a given structural disturbance affects both reduced form disturbances. For example, a random upward shift in the demand function, caused by a positive value of u_d, leads to an upward increase in price ($v_1 > 0$) and quantity ($v_2 > 0$). As an exercise, you should try to prove that

$$Cov[v_1, v_2] = E[v_1 v_2] = E[(\frac{u_d - u_s}{\beta_1 - \alpha_1})(\frac{\beta_1 u_d - \alpha_1 u_s}{\beta_1 - \alpha_1})]$$

$$= \frac{\beta_1 V[u_d] - (\beta_1 + \alpha_1) Cov[u_d, u_s] + \alpha_1 V[u_s]}{(\beta_1 - \alpha_1)^2}$$

This covariance could be either positive or negative.

11.6. Matrix representation of a structural equation model

The algebra we used in the previous section to get from the structural equation models to the reduced form of the models is quite easy, because our models only have two equations and two endogenous variables. But imagine the mess we would get into if there were ten equations and ten endogenous variables! A large-scale econometric model could easily involve 1000 equations and 1000 endogenous variables! We need an alternative way of thinking about the algebra involved in econometrics which is just as applicable to a model with 1000 equations as it is to a model with only two equations. This is why matrix algebra is so important to econometricians.

Let us go back to the two-equation Keynesian national income model of the previous section. We can write it in matrix form as:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ I_t \end{bmatrix} + \begin{bmatrix} u_t \\ 0 \end{bmatrix}$$

You can check by multiplying out the matrices that this gives us our two structural equations. Let

$$\mathbf{y}_t \equiv \begin{bmatrix} C_t \\ Y_t \end{bmatrix} \quad \mathbf{B} \equiv \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix} \quad \mathbf{G} \equiv \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{x}_t \equiv \begin{bmatrix} 1 \\ I_t \end{bmatrix} \quad \mathbf{u}_t \equiv \begin{bmatrix} u_t \\ 0 \end{bmatrix}$$

Then we can write our two-equation model very concisely in matrix form as

$$\mathbf{y}_t = \mathbf{B}\mathbf{y}_t + \mathbf{G}\mathbf{x}_t + \mathbf{u}_t$$

But we can also use this matrix equation to represent any structural equation model we want, even one involving 1000 equations and 1000 endogenous variables, simply by defining $\mathbf{y}_t$, $\mathbf{B}$, $\mathbf{G}$, $\mathbf{x}_t$ and $\mathbf{u}_t$ appropriately. So if we use this matrix equation to do our algebra, the results we get will be just as applicable to a model with 1000 equations as it is to a model with only two equations.

Let us use the concise matrix version of the structural equation model to derive the matrix version of the reduced form model. First, we take the term $\mathbf{B}\mathbf{y}_t$ over to the left hand side, and write

$$\mathbf{y}_t - \mathbf{B}\mathbf{y}_t = \mathbf{G}\mathbf{x}_t + \mathbf{u}_t$$

But $\mathbf{y}_t - \mathbf{B}\mathbf{y}_t = (\mathbf{I}-\mathbf{B})\mathbf{y}_t$, where $\mathbf{I}$ is a conformable identity matrix, so we can write

$$(\mathbf{I}-\mathbf{B})\mathbf{y}_t = \mathbf{G}\mathbf{x}_t + \mathbf{u}_t$$

Now, provided that $(\mathbf{I}-\mathbf{B})$ is a square matrix with linearly independent rows and columns, it will have an inverse $(\mathbf{I}-\mathbf{B})^{-1}$. This will always be the case when there are as many independent equations in our structural model as there are endogenous variables. We can then multiply both sides of the last equation by this inverse to get

$$\mathbf{y}_t = (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{G}\mathbf{x}_t + (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}_t$$

And this is the reduced form of the matrix version of the structural equation model! We have the endogenous variables in a vector $\mathbf{y}_t$ on the left hand side of the equation, and the right hand side involves only parameters, the observed exogenous variables and the disturbances. Just as in the two-equation case, we can define new parameters for the reduced form of the model by letting $\Pi \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{G}$ and we can then write the reduced form of the model as

$$\mathbf{y}_t = \Pi\mathbf{x}_t + \mathbf{v}_t$$

where $\mathbf{v}_t \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}_t$. We can easily prove that this agrees with the reduced form equations we got for the two-equation model in the previous section by writing out the relevant matrix products in full. We have

$$(\mathbf{I}\text{-}\mathbf{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ -1 & 1 \end{bmatrix} \quad \text{so} \quad (\mathbf{I}\text{-}\mathbf{B})^{-1} = \begin{bmatrix} 1 & -\beta \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{1-\beta}\begin{bmatrix} 1 & \beta \\ 1 & 1 \end{bmatrix}$$

Then

$$\Pi\mathbf{x}_t = (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{G}\mathbf{x}_t = \frac{1}{1-\beta}\begin{bmatrix} 1 & \beta \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ I_t \end{bmatrix} = \frac{1}{1-\beta}\begin{bmatrix} \alpha + \beta I_t \\ \alpha + I_t \end{bmatrix}$$

and

$$\mathbf{v}_t \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}_t = \frac{1}{1-\beta}\begin{bmatrix} 1 & \beta \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_t \\ 0 \end{bmatrix} = \frac{1}{1-\beta}\begin{bmatrix} u_t \\ u_t \end{bmatrix}$$

So using the matrix version of the reduced form of the model we get

$$\mathbf{y}_t = \Pi\mathbf{x}_t + \mathbf{v}_t = \frac{1}{1-\beta}\begin{bmatrix} \alpha + \beta I_t \\ \alpha + I_t \end{bmatrix} + \frac{1}{1-\beta}\begin{bmatrix} u_t \\ u_t \end{bmatrix} = \begin{bmatrix} C_t \\ Y_t \end{bmatrix}$$

which is obviously the same as what we got in the previous section.

11.7. <u>A worked example of another two-equation structural model</u>

The following problem is typical of the sort of thing you might be asked to tackle as part of an exam question. I shall work through it in full in this section, to show you the sort of thing I want to see, and then I'll give you something similar to do in an assignment for next week. Just follow exactly the same procedure in your homework.

Consider the following model:

$$Y_i = \beta_0 + \beta_1 H_i + \beta_2 A_i + \beta_3 E_i + u_{1i}$$
$$H_i = \delta_0 + \delta_1 Y_i + \delta_2 A_i + \delta_3 E_i + u_{2i}$$

endogenous variables: Y_i , H_i ; *exogenous variables:* A_i , E_i.

(i). Set up this model in matrix terms in the form $\mathbf{y}_i = \mathbf{B}\mathbf{y}_i + \mathbf{G}\mathbf{x}_i + \mathbf{u}_i$ *, where* $\mathbf{y}_i$ *is a vector of endogenous variables,* $\mathbf{x}_i$ *is a vector of exogenous variables,* $\mathbf{B}$ *and* $\mathbf{G}$ *are coefficient matrices, and* $\mathbf{u}_i$ *is a vector of disturbances.*

(ii). Find $(\mathbf{I}\text{-}\mathbf{B})^{-1}$ *and then carry out the matrix multiplication* $\Pi\mathbf{x}_i \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{G}\mathbf{x}_i$ *(your final answer should be a 2×1 matrix).*

(iii). Carry out the matrix multiplication $\mathbf{v}_i \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}_i$ *(your final answer should be a 2×1 matrix) and use your answer together with your answer to part (ii) to write the reduced form of the model as two <u>separate</u> reduced form equations (i.e. <u>not</u> in matrix form).*

Solution:

(i).
$$\begin{bmatrix} Y_i \\ H_i \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 \\ \delta_1 & 0 \end{bmatrix}\begin{bmatrix} Y_i \\ H_i \end{bmatrix} + \begin{bmatrix} \beta_0 & \beta_2 & \beta_3 \\ \delta_0 & \delta_2 & \delta_3 \end{bmatrix}\begin{bmatrix} 1 \\ A_i \\ E_i \end{bmatrix} + \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix}$$

(ii). $(\mathbf{I}\text{-}\mathbf{B})^{-1} = \begin{bmatrix} 1 & -\beta_1 \\ -\delta_1 & 1 \end{bmatrix}^{-1} = \dfrac{1}{1-\delta_1\beta_1}\begin{bmatrix} 1 & \beta_1 \\ \delta_1 & 1 \end{bmatrix}$

so we have $\Pi\mathbf{x}_i = (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{G}\mathbf{x}_i = \dfrac{1}{1-\delta_1\beta_1}\begin{bmatrix} 1 & \beta_1 \\ \delta_1 & 1 \end{bmatrix}\begin{bmatrix} \beta_0 & \beta_2 & \beta_3 \\ \delta_0 & \delta_2 & \delta_3 \end{bmatrix}\begin{bmatrix} 1 \\ A_i \\ E_i \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{\beta_0+\delta_0\beta_1}{1-\delta_1\beta_1} + \dfrac{\beta_2+\delta_2\beta_1}{1-\delta_1\beta_1}A_i + \dfrac{\beta_3+\delta_3\beta_1}{1-\delta_1\beta_1}E_i \\ \dfrac{\delta_0+\delta_1\beta_0}{1-\delta_1\beta_1} + \dfrac{\delta_2+\delta_1\beta_2}{1-\delta_1\beta_1}A_i + \dfrac{\delta_3+\delta_1\beta_3}{1-\delta_1\beta_1}E_i \end{bmatrix}$$

(iii). $\mathbf{v}_i = (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}_i = \dfrac{1}{1-\delta_1\beta_1}\begin{bmatrix} 1 & \beta_1 \\ \delta_1 & 1 \end{bmatrix}\begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} = \begin{bmatrix} \dfrac{u_{1i}+\beta_1 u_{2i}}{1-\delta_1\beta_1} \\ \dfrac{\delta_1 u_{1i}+u_{2i}}{1-\delta_1\beta_1} \end{bmatrix}$

so, using the answer to part (ii), the two separate equations of the reduced form model are:

$$Y_i = \frac{\beta_0+\delta_0\beta_1}{1-\delta_1\beta_1} + \frac{\beta_2+\delta_2\beta_1}{1-\delta_1\beta_1}A_i + \frac{\beta_3+\delta_3\beta_1}{1-\delta_1\beta_1}E_i + \frac{u_{1i}+\beta_1 u_{2i}}{1-\delta_1\beta_1}$$
$$H_i = \frac{\delta_0+\delta_1\beta_0}{1-\delta_1\beta_1} + \frac{\delta_2+\delta_1\beta_2}{1-\delta_1\beta_1}A_i + \frac{\delta_3+\delta_1\beta_3}{1-\delta_1\beta_1}E_i + \frac{\delta_1 u_{1i}+u_{2i}}{1-\delta_1\beta_1}$$

(End of Lecture 11)

EC2203 QUANTITATIVE METHODS IN ECONOMICS II (1998-1999)

Assignment for Lecture 11. Introduction to econometric models

Question

Consider the demand and supply model from the Lecture:

$$Q = \alpha_0 + \alpha_1 P + u_d$$
$$Q = \beta_0 + \beta_1(P-E) + u_s$$

endogenous variables: Q, P

exogenous variable: E

(a). Set up this model in matrix terms in the form $\mathbf{y} = \mathbf{By} + \mathbf{Gx} + \mathbf{u}$ *, where* $\mathbf{y}$ *is a vector of endogenous variables,* $\mathbf{x}$ *is a vector of exogenous variables,* $\mathbf{B}$ *and* $\mathbf{G}$ *are coefficient matrices, and* $\mathbf{u}$ *is a vector of disturbances (Hint: solve the first equation for P first).*

(b). Find $(\mathbf{I}\text{-}\mathbf{B})^{-1}$ *and then carry out the matrix multiplication* $\Pi\mathbf{x} \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{Gx}$ *(your final answer should be a 2×1 matrix).*

(c). Carry out the matrix multiplication $\mathbf{v} \equiv (\mathbf{I}\text{-}\mathbf{B})^{-1}\mathbf{u}$ *(your final answer should be a 2×1 matrix) and use your answer together with your answer to part (ii) to write the reduced form of the model as two separate reduced form equations (i.e. not in matrix form). Check that these are the same as those given in the Lecture.*

(d). Using the symbols in the Lecture, the expressions for the reduced form coefficients in terms of the structural parameters are

$$\gamma_0 = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \qquad \gamma_1 = \frac{\beta_1}{\beta_1 - \alpha_1} \qquad \delta_0 = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\beta_1 - \alpha_1} \qquad \delta_1 = \frac{\alpha_1\beta_1}{\beta_1 - \alpha_1}$$

It was asserted in the Lecture that we can solve back from the reduced form coefficients to the structural parameters, to get the algebraic solution

$$\alpha_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1} \qquad \alpha_1 = \frac{\delta_1}{\gamma_1} \qquad \beta_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1 - 1} \qquad \beta_1 = \frac{\delta_1}{\gamma_1 - 1}$$

Showing your workings in full, derive these formulae from the four expressions for the reduced form coefficients.

SOLUTIONS

(a) We can solve the first equation for P, and rewrite the model as

"Inverse demand function": $P = \frac{1}{\alpha_1} Q - \frac{\alpha_0}{\alpha_1} - \frac{1}{\alpha_1} u_d$

supply function : $Q = \beta_0 + \beta_1 (P - \epsilon) + u_s$

In matrix terms:

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \beta_1 & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} + \begin{bmatrix} -\frac{\alpha_0}{\alpha_1} & 0 \\ \beta_0 & -\beta_1 \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} + \begin{bmatrix} -\frac{1}{\alpha_1} u_d \\ u_s \end{bmatrix}$$

$$y = B \cdot y + G \cdot x + u$$

(b) $$(I - B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \beta_1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/\alpha_1 \\ -\beta_1 & 1 \end{bmatrix}$$

The determinant of $(I - B)$ is $1 - \frac{\beta_1}{\alpha_1} = \frac{\alpha_1 - \beta_1}{\alpha_1}$

$$= -\frac{(\beta_1 - \alpha_1)}{\alpha_1}$$

Therefore $$(I - B)^{-1} = -\left(\frac{\alpha_1}{\beta_1 - \alpha_1}\right) \begin{bmatrix} 1 & 1/\alpha_1 \\ \beta_1 & 1 \end{bmatrix}$$

To find $\Pi x = (I - B)^{-1} G x$, proceed in stages. First compute Gx:

$$Gx = \begin{bmatrix} -\frac{\alpha_0}{\alpha_1} & 0 \\ \beta_0 & -\beta_1 \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_0}{\alpha_1} \\ \beta_0 - \beta_1 \epsilon \end{bmatrix}$$

(2×2) $\quad$ (2×1) $\quad$ (2×1)

Then:

$$\Pi x = (I-B)^{-1} G x = -\left(\frac{\alpha_1}{\beta_1-\alpha_1}\right)\begin{bmatrix} 1 & 1/\alpha_1 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_0/\alpha_1 \\ \beta_0 - \beta_1 E \end{bmatrix}$$

$$= -\left(\frac{\alpha_1}{\beta_1-\alpha_1}\right)\begin{bmatrix} -\alpha_0/\alpha_1 + \frac{\beta_0-\beta_1 E}{\alpha_1} \\ -\frac{\beta_1\alpha_0}{\alpha_1} + (\beta_0-\beta_1 E) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_0}{\beta_1-\alpha_1} + \frac{\beta_1 E-\beta_0}{\beta_1-\alpha_1} \\ \frac{\beta_1\alpha_0}{\beta_1-\alpha_1} + \frac{\alpha_1\beta_1 E-\alpha_1\beta_0}{\beta_1-\alpha_1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_0-\beta_0}{\beta_1-\alpha_1} + \frac{\beta_1}{\beta_1-\alpha_1}E \\ \frac{\beta_1\alpha_0-\alpha_1\beta_0}{\beta_1-\alpha_1} + \frac{\alpha_1\beta_1}{\beta_1-\alpha_1}E \end{bmatrix}$$

(c) $$v = (I-B)^{-1}u = -\left(\frac{\alpha_1}{\beta_1-\alpha_1}\right)\begin{bmatrix} 1 & 1/\alpha_1 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\alpha_1}u_d \\ u_s \end{bmatrix}$$

$$= -\left(\frac{\alpha_1}{\beta_1-\alpha_1}\right)\begin{bmatrix} \frac{u_s-u_d}{\alpha_1} \\ u_s - \frac{\beta_1}{\alpha_1}u_d \end{bmatrix} = \begin{bmatrix} \frac{u_d-u_s}{\beta_1-\alpha_1} \\ \frac{\beta_1 u_d-\alpha_1 u_s}{\beta_1-\alpha_1} \end{bmatrix}$$

The reduced form model in matrix terms is

$$y = \Pi x + v$$

Looking at the corresponding rows of y, Πx and v, we have

$$P = \frac{\alpha_0-\beta_0}{\beta_1-\alpha_1} + \frac{\beta_1}{\beta_1-\alpha_1}E + \frac{u_d-u_s}{\beta_1-\alpha_1}$$

$$Q = \frac{\beta_1\alpha_0-\alpha_1\beta_0}{\beta_1-\alpha_1} + \frac{\alpha_1\beta_1}{\beta_1-\alpha_1}E + \frac{\beta_1 u_d-\alpha_1 u_s}{\beta_1-\alpha_1}$$

These are the same as the equations given on page 10 of the handout for Lecture 11.

(d) We can solve for α_1 by substituting $\gamma_1 = \frac{\beta_1}{\beta_1 - \alpha_1}$ into the equation for δ_1 to get

$$\delta_1 = \alpha_1 \gamma_1 \Rightarrow \boxed{\alpha_1 = \frac{\delta_1}{\gamma_1}} \quad (A)$$

We can solve for β_1 by substituting (A) back into the equation for γ_1:

$$\gamma_1 = \frac{\beta_1}{\beta_1 - \alpha_1} = \frac{\beta_1}{\beta_1 - \frac{\delta_1}{\gamma_1}} = \frac{\beta_1}{\frac{\gamma_1\beta_1 - \delta_1}{\gamma_1}}$$

$$= \frac{\gamma_1\beta_1}{\gamma_1\beta_1 - \delta_1}$$

Thus, $(\gamma_1\beta_1 - \delta_1)\gamma_1 = \gamma_1\beta_1$

$$\Rightarrow \gamma_1\beta_1 - \delta_1 = \beta_1$$

$$\Rightarrow (\gamma_1 - 1)\beta_1 = \delta_1 \Rightarrow \boxed{\beta_1 = \frac{\delta_1}{\gamma_1 - 1}} \quad (B)$$

We now solve for β_0 by using the equations for γ_0 and δ_0. We have

$$\gamma_0 = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \Rightarrow (\beta_1 - \alpha_1)\gamma_0 = \alpha_0 - \beta_0$$

$$\Rightarrow \boxed{\alpha_0 = (\beta_1 - \alpha_1)\gamma_0 + \beta_0} \quad (C)$$

We also have

$$\delta_0 = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\beta_1 - \alpha_1} \Rightarrow \boxed{(\beta_1 - \alpha_1)\delta_0 = \alpha_0\beta_1 - \alpha_1\beta_0} \quad (D)$$

Substituting (C) into (D) we get

$$(\beta_1-\alpha_1)\delta_0 = \{(\beta_1-\alpha_1)\gamma_0 + \beta_0\}\beta_1 - \alpha_1\beta_0$$

$$= (\beta_1-\alpha_1)\gamma_0\beta_1 + \beta_0\beta_1 - \beta_0\alpha_1$$

$$= (\beta_1-\alpha_1)\gamma_0\beta_1 + (\beta_1-\alpha_1)\beta_0$$

Dividing both sides by $(\beta_1-\alpha_1)$ we get

$$\delta_0 = \gamma_0\beta_1 + \beta_0 \Rightarrow \boxed{\beta_0 = \delta_0 - \gamma_0\beta_1} \quad (E)$$

Substituting (B) into (E) gives

$$\boxed{\beta_0 = \delta_0 - \frac{\gamma_0\delta_1}{\gamma_1 - 1}} \quad (F)$$

Finally, we get α_0 by substituting (F) into (C). Note first that

$$(\beta_1-\alpha_1) = \frac{\delta_1}{\gamma_1-1} - \frac{\delta_1}{\gamma_1} = \frac{\gamma_1\delta_1-(\gamma_1-1)\delta_1}{\gamma_1(\gamma_1-1)} = \frac{\delta_1}{\gamma_1(\gamma_1-1)}$$

Substituting this result, and (F), into (C) gives

$$\alpha_0 = \frac{\delta_1\gamma_0}{\gamma_1(\gamma_1-1)} + \delta_0 - \frac{\gamma_0\delta_1}{\gamma_1-1}$$

$$= \delta_0 + \frac{\delta_1\gamma_0 - \gamma_0\delta_1\gamma_1}{\gamma_1(\gamma_1-1)} = \delta_0 + \frac{\delta_1\gamma_0(1-\gamma_1)}{\gamma_1(\gamma_1-1)}$$

$$= \delta_0 - \frac{\delta_1\gamma_0\cancel{(\gamma_1-1)}}{\gamma_1\cancel{(\gamma_1-1)}} = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1}$$