(i)

Lecture 12. Identification

12.1. The identification problem

In the last lecture, we used the following simple two-equation structural model to introduce some basic concepts:

$$Q = \alpha_0 + \alpha_1 P + u_d$$
$$Q = \beta_0 + \beta_1(P-E) + u_s$$

endogenous variables: Q, P

exogenous variable: E

Recall that the reduced form of the model is

$$P = \gamma_0 + \gamma_1 E + v_1$$
$$Q = \delta_0 + \delta_1 E + v_2$$

We said that these reduced form equations can be estimated from data to get the parameter estimates γ_0^*, γ_1^*, δ_0^*, δ_1^*. These can then be used in conjunction with the following formulae to get estimates for the structural parameters:

$$\alpha_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1} \qquad \alpha_1 = \frac{\delta_1}{\gamma_1} \qquad \beta_0 = \delta_0 - \frac{\delta_1\gamma_0}{\gamma_1 - 1} \qquad \beta_1 = \frac{\delta_1}{\gamma_1 - 1}$$

Think of the reduced form parameter estimates γ_0^*, γ_1^*, δ_0^*, δ_1^* as visible pieces of information that we have managed to 'squeeze out' of our data by estimating the reduced form equations of the model. Clearly, we will not be able to estimate a structural parameter unless it can be expressed as a function only of visible pieces of information like these. When faced with a particular structural equation model, the identification problem is simply this: *can the reduced form parameters be used to deduce unique values of the structural parameters ?*

A structural equation model is said to be not identified if we are not able to solve out all the structural parameters appearing in the model in terms of the parameters of the reduced form equations. We may be able to solve out some, or even most of the structural parameters, but unless we can solve out all the structural parameters, we say that the structural model is 'not identified'.

A structural equation model is said to be just identified if there is only one way to solve out all the structural parameters that appear in the model. Our simple two-equation model above is an example of a just identified model, since the expression for each structural parameter in terms of the reduced form parameters is unique: it is impossible to obtain two or more different estimates of a structural parameter by combining reduced form parameters in different ways.

Finally, a structural equation model is said to be over identified if there is at least one way to solve out all the structural parameters in terms of the reduced form parameters, and more than one way to do so for at least one of the structural parameters. In other words, there is at least one structural parameter for which we can produce different estimates by combining the reduced form parameters in different ways, and there is no way of choosing between one particular estimate and another. As might be expected, this gives rise to certain problems which we shall talk about in a later lecture.

We can apply the same basic definitions of 'not identified', 'just identified', and 'over identified' to individual equations, and even to individual parameters. So we can speak of a structural equation as being not identified, for example, if we cannot solve out all the parameters that appear in that equation. It may seem a bit strange, but we could in theory be faced with a structural equation model in which one equation was not identified, one just identified, and one over identified! In that case, of course, the model as a whole would be 'not identified', even though some equations are just identified, and others are over identified. Similarly, we can speak of an individual structural parameter as being not identified, just identified or over identified depending on whether (and in how many different ways) we can solve it out in terms of reduced form parameters.

12.2. Exclusion restrictions and homogeneous linear restrictions

A structural equation model is always 'not identified' if there are more structural parameters than there are reduced form parameters. This is because there are not enough reduced form parameters 'to go round', so some structural parameters cannot be solved out in a unique way in terms of reduced form parameters alone. ***A necessary condition for identification is that there be at least as many reduced form parameters as there are structural parameters, but this is not a sufficient condition***. In other words, a structural equation model may be 'not identified' even when there are just as many reduced form parameters as there are structural parameters. We shall provide necessary *and* sufficient conditions for identification in the next section.

In this section, we shall use the matrix representation of the general structural equation model to show that the most general form of a structural equation model (ie. one with a full complement of unrestricted parameters) is always 'not identified'. We shall then see how identification can be obtained by imposing certain types of restrictions on the structural parameters.

Suppose there are T equations, T endogenous variables and K exogenous variables in the structural model. The T endogenous variables are denoted by y_{1i}, y_{2i}, ..., y_{Ti} and the K exogenous variables are denoted by x_{1i}, x_{2i}, ..., x_{Ki}. The structural parameters are the coefficients appearing in the T equations, coefficients of endogenous and exogenous variables being denoted by β's and γ's respectively. Each structural parameter is written with two subscripts, the first indicating the equation in which the parameter appears, and the second indicating the variable to which the coefficient is attached. For each individual i = 1, 2, ..., N, we can write out the T structural equations as follows:

$$
\begin{aligned}
y_{1i} &= \beta_{11}y_{1i} + \beta_{12}y_{2i} + \cdots + \beta_{1T}y_{Ti} + \gamma_{11}x_{1i} + \gamma_{12}x_{2i} + \cdots + \gamma_{1K}x_{Ki} + u_{1i} \\
y_{2i} &= \beta_{21}y_{1i} + \beta_{22}y_{2i} + \cdots + \beta_{2T}y_{Ti} + \gamma_{21}x_{1i} + \gamma_{22}x_{2i} + \cdots + \gamma_{2K}x_{Ki} + u_{2i} \\
&\vdots \\
y_{Ti} &= \beta_{T1}y_{1i} + \beta_{T2}y_{2i} + \cdots + \beta_{TT}y_{Ti} + \gamma_{T1}x_{1i} + \gamma_{T2}x_{2i} + \cdots + \gamma_{TK}x_{Ki} + u_{Ti}
\end{aligned}
$$

The model can be written in matrix terms as

$$
\underset{T\times 1}{\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix}} = \underset{T\times T}{\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1T} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{T1} & \beta_{T2} & \cdots & \beta_{TT} \end{bmatrix}} \underset{T\times 1}{\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix}} + \underset{T\times K}{\begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1K} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T1} & \gamma_{T2} & \cdots & \gamma_{TK} \end{bmatrix}} \underset{K\times 1}{\begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{Ki} \end{bmatrix}} + \underset{T\times 1}{\begin{bmatrix} u_{1i} \\ u_{2i} \\ \vdots \\ u_{Ti} \end{bmatrix}}
$$

There are two points to make at this stage. Firstly, a structural equation is never written in such a way that the *same* endogenous variable appears both on the left hand side and the right hand side of the equation. Thus, the elements along the main diagonal of the coefficient matrix of the endogenous variables must all equal zero, otherwise the dependent variable on the left hand side would also appear on the right. Secondly, the first exogenous variable in each equation is always a dummy variable taking the value 1, so that its coefficient γ_{t1} represents a constant term in equation t. Thus, a general structural equation model will have the following matrix representation:

$$\underset{T\times 1}{\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix}} = \underset{T\times T}{\begin{bmatrix} 0 & \beta_{12} & \cdots & \beta_{1T} \\ \beta_{21} & 0 & \cdots & \beta_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{T1} & \beta_{T2} & \cdots & 0 \end{bmatrix}} \underset{T\times 1}{\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix}} + \underset{T\times K}{\begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1K} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T1} & \gamma_{T2} & \cdots & \gamma_{TK} \end{bmatrix}} \underset{K\times 1}{\begin{bmatrix} 1 \\ x_{2i} \\ \vdots \\ x_{Ki} \end{bmatrix}} + \underset{T\times 1}{\begin{bmatrix} u_{1i} \\ u_{2i} \\ \vdots \\ u_{Ti} \end{bmatrix}}$$

Using our notation from the last lecture, this can be written more concisely as

$$\mathbf{y}_i = \mathbf{B}\mathbf{y}_i + \mathbf{G}\mathbf{x}_i + \mathbf{u}_i$$

Next, consider the reduced form of the model. The general reduced form of T equations, each expressing an endogenous variable in terms of exogenous variables and disturbances, can be written as

$$\begin{aligned} y_{1i} &= \pi_{11} + \pi_{12}x_{2i} + \cdots + \pi_{1K}x_{Ki} + v_{1i} \\ y_{2i} &= \pi_{21} + \pi_{22}x_{2i} + \cdots + \pi_{2K}x_{Ki} + v_{2i} \\ &\cdots \\ &\cdots \\ &\cdots \\ y_{Ti} &= \pi_{T1} + \pi_{T2}x_{2i} + \cdots + \pi_{TK}x_{Ki} + v_{Ti} \end{aligned}$$

(note that x_{1i} has been set equal to 1 in each equation). The π's represent the reduced form coefficients, and the v's the reduced form disturbances. Again, these equations can be written in matrix terms as

$$\underset{T\times 1}{\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix}} = \underset{T\times K}{\begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1K} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{T1} & \pi_{T2} & \cdots & \pi_{TK} \end{bmatrix}} \underset{K\times 1}{\begin{bmatrix} 1 \\ x_{2i} \\ \vdots \\ x_{Ki} \end{bmatrix}} + \underset{T\times 1}{\begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{Ti} \end{bmatrix}}$$

We can use our notation from the previous lecture to write this in more concise form as

$$\mathbf{y}_i = \mathbf{\Pi}\mathbf{x}_i + \mathbf{v}_i$$

We saw in the last lecture that the relation between the structural form and the reduced form can be obtained explicitly by solving the structural form as follows:

$$\mathbf{y}_i = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{G}\mathbf{x}_i + (\mathbf{I}-\mathbf{B})^{-1}\mathbf{u}_i$$

Thus, we can write the reduced form coefficient matrix in terms of the coefficient matrices of the structural equation model as

$$\Pi = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{G}$$

where Π is a T×K matrix, $(\mathbf{I}-\mathbf{B})^{-1}$ is a T×T matrix, and $\mathbf{G}$ is a T×K matrix. Now, we can estimate the TK parameters contained in Π to get TK 'visible pieces of information'. But on the right hand side there are T(T-1) structural parameters in $(\mathbf{I}-\mathbf{B})^{-1}$, and TK structural parameters in $\mathbf{G}$. Thus, there is a total of T(T-1) + TK structural parameters to be solved out in terms of only TK reduced form parameters. ***Since there are more structural parameters than reduced form parameters in the general structural equation model, the general model is always 'not identified' when there are no restrictions imposed on the parameters contained in the matrix (I-B)$^{-1}$G.***

There are two basic types of restriction we might impose on the structural parameters in **(I-B)$^{-1}$G** to make the structural equation model identified: exclusion restrictions, which simply involve setting elements of **B** and **G** equal to zero (guided as far as possible by theoretical economic considerations), and homogeneous linear restrictions, which involve setting linear combinations of structural parameters in **B** and **G** equal to zero. To illustrate the use of these restrictions, we can employ the following two-equation model:

$$y_{1i} = \beta_{12}y_{2i} + \gamma_{11} + \gamma_{12}x_{2i} + u_{1i}$$
$$y_{2i} = \beta_{21}y_{1i} + \gamma_{21} + \gamma_{22}x_{2i} + u_{2i}$$

This is a general model of the type considered above, with no restrictions imposed on **B** or **G**. It can be written in matrix form as

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{bmatrix}\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} 1 \\ x_{2i} \end{bmatrix} + \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix}$$

Now, if we write out the matrices of the equation Π = **(I-B)$^{-1}$G** in full, we get

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\gamma_{11} + \beta_{12}\gamma_{21}}{1-\beta_{12}\beta_{21}} & \dfrac{\gamma_{12} + \beta_{12}\gamma_{22}}{1-\beta_{12}\beta_{21}} \\ \dfrac{\beta_{21}\gamma_{11} + \gamma_{21}}{1-\beta_{12}\beta_{21}} & \dfrac{\beta_{21}\gamma_{12} + \gamma_{22}}{1-\beta_{12}\beta_{21}} \end{bmatrix} = (\mathbf{I}-\mathbf{B})^{-1}\mathbf{G}$$

The matrix on the right hand side involves six different structural parameters, but the matrix on the left can only provide us with four visible pieces of information. Thus, as it currently stands, the structural equation model must be 'not identified'. But suppose we impose the following exclusion restrictions: $\gamma_{11} = 0$, $\gamma_{12} = 0$, $\gamma_{22} = 0$, and the following homogeneous linear restriction: $\beta_{21} + \gamma_{21} = 0$. The homogeneous linear restriction implies that $\beta_{21} = -\gamma_{21}$, so the above matrix equation becomes

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\beta_{12}\gamma_{21}}{1+\beta_{12}\gamma_{21}} & 0 \\ \dfrac{\gamma_{21}}{1+\beta_{12}\gamma_{21}} & 0 \end{bmatrix}$$

Now there are only two different structural parameters involved in the matrix on the right hand side. By inspection, $\pi_{11} = (\beta_{12}\gamma_{21})/(1+\beta_{12}\gamma_{21})$ and $\pi_{21} = \gamma_{21}/(1+\beta_{12}\gamma_{21})$, so we can solve for the structural parameters uniquely in terms of reduced form parameters as follows: $\beta_{12} = \pi_{11}/\pi_{21}$ and $\gamma_{21} = \pi_{21}/(1-\pi_{11})$. Thus, the three exclusion restrictions and the single homogeneous linear restriction have turned the unidentified general model into the following just identified model:

$$y_{1i} = \beta_{12}y_{2i} + u_{1i}$$
$$y_{2i} = -\gamma_{21}y_{1i} + \gamma_{21} + u_{2i}$$

12.3. Rank and order conditions for identification

We have seen that an unidentified structural equation model can be turned into an identified model by imposing restrictions on the structural parameters. What we would now like is a set of easily verifiable conditions that determine unambiguously whether a particular structural equation model is identified. The basic approach is to assess the identifiability of each structural equation individually.

In this section, we shall consider a *necessary* condition for the identification of a structural equation called the order condition, and a *necessary and sufficient* condition called the rank condition. The reason we need both of these is that the order condition is always very easy to apply, but the rank condition can be difficult to apply, especially in large models. Most practitioners just apply the order condition and 'hope' that the rank condition is satisfied. It often is, but it is best to check the rank condition if at all possible.

It was stated above that the basic approach is to assess the identifiability of each structural equation individually. The first structural equation in the unrestricted model of the previous section is

$$y_{1i} = \beta_{12}y_{2i} + \beta_{13}y_{3i} + \cdots + \beta_{1T}y_{Ti} + \gamma_{11} + \gamma_{12}x_{2i} + \cdots + \gamma_{1K}x_{Ki} + u_{1i}$$

As it stands, this equation involves (T-1) + K structural parameters. The associated reduced form equation is

$$y_{1i} = \pi_{11} + \pi_{12}x_{2i} + \cdots + \pi_{1K}x_{Ki} + v_{1i}$$

which involves K estimable reduced form parameters (ie. K potentially 'visible' pieces of information). So we have K 'known' parameters (the π's) and (T-1) + K unknown parameters (the β's and the γ's). To identify the structural equation we need more information, and we get it from economic theory. Economic theory gives us restrictions eg. it might suggest that some variables do not appear in the structural equation so that their coefficients are zero, or that some coefficients when added together sum to one or zero. This reduces the number of parameters to be estimated, as we saw in the previous section. Since a necessary condition for identification is that the number of reduced form parameters is at least as large as the number of structural parameters to be estimated, the number of restrictions we need in the first equation (denoted by d_1) must satisfy

$$d_1 + K \geq (T - 1) + K$$

which is equivalent to $\quad d_1 \geq (T-1)$

This is the order condition: ***in a model consisting of T structural equations, an equation is identified only if the number of parameter restrictions incorporated in it is greater than or equal to (T-1).*** So if a structural model consists of four equations, each individual equation must incorporate at least three exclusion restrictions and/or homogeneous linear restrictions if it is to be identified. If a structural model consists of ten equations, each individual equation must incorporate at least nine restrictions etc.. If a structural equation does not satisfy this condition, it cannot be identified. If it does satisfy this condition, it may or may not be identified, and a further test (ie. the rank condition) is required to make absolutely sure. We shall look in detail at an example of how to apply the rank and order conditions, but let us first state the rank condition, which is a little bit more involved. The rank condition is as follows: ***if only exclusion restrictions are incorporated in a model consisting of T structural equations, an equation is identified if and only if at least one non-zero (T-1)×(T-1) determinant is contained in the array of coefficients with which those variables excluded from the equation in question appear in the other equations. If homogeneous linear restrictions are also incorporated, we again seek a non-zero (T-1)×(T-1) determinant in an array of coefficients, but now we obtain the array by applying the homogeneous linear restriction to the other equations.*** This sounds very complicated, but it is actually very easy to apply in the simple examples we shall deal with in this course. We shall go through some simple examples in the next section. If the rank condition is satisfied, the order condition is automatically satisfied, but not vice versa, so the order condition is necessary but not sufficient. (Recall that the rank of a matrix is the order of the largest non-zero determinant that it contains).

12.4. Applying the rank and order conditions in practice: some worked examples

In order to apply the rank and order conditions, we follow the following steps. (You will not be required to deal with homogenous linear restrictions in this course, so the procedure described below assumes that only exclusion restrictions are being considered. However, we will look at a simple example involving a homogeneous linear restriction in case you come across this in your later studies).

Step I. First, write the structural equation model in matrix terms in the form $\mathbf{y}_i = \mathbf{B}\mathbf{y}_i + \mathbf{G}\mathbf{x}_i + \mathbf{u}_i$.
Step II. Second, form the matrix $\mathbf{A} = [(\mathbf{I}\text{-}\mathbf{B})|(\text{-}\mathbf{G})]$ (see the examples below).
Step III. To apply the order condition to the ith equation, count the number of zeros in the ith row of A. This gives the number of exclusion restrictions incorporated in the ith equation. If the number of zeros is less than T-1, the equation is not identified. If the number of zeros equals T-1, the equation is just identified by the order condition. If the number of zeros is greater than T-1, the equation is over identified by the order condition.
Step IV. The rank condition is applied only to those equations which are just identified or over identified by the order condition. To apply the rank condition to the ith equation, first see what variables (either endogenous or exogenous) are excluded from it. Then look at the coefficients of those excluded variables in the remaining equations. An array of these coefficients will have T-1 rows, and as many columns as there are excluded variables. If this array has at least one non-zero $(T\text{-}1)\times(T\text{-}1)$ determinant, the ith equation is identified. Otherwise, the equation is not identified, even if the order condition indicates that it is just identified or over identified.

Note: Do not apply the rank and order conditions to identities, such as $Y_t = C_t + I_t + G_t$. Equations like these do not involve unknown parameters, so no question of identification arises. However, you must include the coefficients of the variables (which are all equal to 1 in the case of $Y_t = C_t + I_t + G_t$) in the matrix $\mathbf{A} = [(\mathbf{I}\text{-}\mathbf{B})|(\text{-}\mathbf{G})]$ *in order to apply the rank and order conditions to the other equations.* See example 2 below.

Example 1:

Assess the identifiability of the following three-equation model by applying the rank and order conditions:

$$y_1 = \alpha_{10} + \alpha_{11}y_2 + \alpha_{12}x_1 + u_1$$
$$y_2 = \beta_{20} + \beta_{21}y_1 + \beta_{22}y_3 + \beta_{23}x_2 + u_2$$
$$y_3 = \gamma_{30} + \gamma_{31}y_2 + \gamma_{32}x_2 + u_3$$

endogenous variables: y_1, y_2, y_3

exogenous variables: x_1, x_2

First, write the model in matrix terms as $y = By + Gx + u$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{11} & 0 \\ \beta_{21} & 0 & \beta_{22} \\ 0 & \gamma_{31} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} \alpha_{10} & \alpha_{12} & 0 \\ \beta_{20} & 0 & \beta_{23} \\ \gamma_{30} & 0 & \gamma_{32} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Then, form the matrix $A = [(I-B)|(-G)]$:

$$A = \begin{bmatrix} 1 & -\alpha_{11} & 0 & -\alpha_{10} & -\alpha_{12} & 0 \\ -\beta_{21} & 1 & -\beta_{22} & -\beta_{20} & 0 & -\beta_{23} \\ 0 & -\gamma_{31} & 1 & -\gamma_{30} & 0 & -\gamma_{32} \end{bmatrix}$$

	No. of zeros	Order condition
1st equation:	2	just identified
2nd equation:	1	not identified
3rd equation:	2	just identified

Now apply the rank condition to the 1st and 3rd equations:

1st equation $\begin{vmatrix} -\beta_{22} & -\beta_{23} \\ 1 & -\gamma_{32} \end{vmatrix} = \beta_{22}\gamma_{32} + \beta_{23} \neq 0$ identified

3rd equation $\begin{vmatrix} 1 & -\alpha_{12} \\ -\beta_{21} & 0 \end{vmatrix} = -\alpha_{12}\beta_{21} \neq 0$ identified

The first and third equations are just identified, but the second equation is not identified.

Example 2:
Assess the identifiability of the following three-equation model of the goods market using the rank and order conditions:

$$C_t = \alpha_0 + \alpha_1 Y_t + \alpha_2 Y_{t-1} + u_1$$
$$I_t = \beta_0 + \beta_1 (Y_t - Y_{t-1}) + u_2$$
$$Y_t = C_t + I_t + G_t$$

endogenous variables: Y_t, C_t, I_t

exogenous variables: G_t

(Note: Y_{t-1} is the lagged value of the endogenous variable, and therefore predetermined at time t. It is treated like an exogeneous variable for the purposes of applying the rank and order conditions).

In matrix terms:

$$\begin{bmatrix} C_t \\ I_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \beta_1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} C_t \\ I_t \\ Y_t \end{bmatrix} + \begin{bmatrix} \alpha_0 & 0 & \alpha_2 \\ \beta_0 & 0 & -\beta_1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ G_t \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

We have

$$[(I-B)|(-G)] = \begin{bmatrix} 1 & 0 & -\alpha_1 & -\alpha_0 & 0 & -\alpha_2 \\ 0 & 1 & -\beta_1 & -\beta_0 & 0 & \beta_1 \\ -1 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

Now apply the order condition to the 1st and 2nd equations:

	No. of zeros/homogeneous linear restrictions	Order condition
1st equation:	2	just identified
2nd equation:	3	over identified

Now apply the rank condition to the 1st and 2nd equations:

1st equation: $\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = -1 \neq 0$ identified

2nd equation: the relevant array of coefficients is

$$\begin{bmatrix} 1 & -\alpha_1 - \alpha_2 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Note that the second column is obtained by applying the analogue of the homogeneous linear restriction $-\beta_1 + \beta_1 = 0$ to the first and third equations. We have eg.

$$\begin{vmatrix} -\alpha_1 - \alpha_2 & 0 \\ 1 & -1 \end{vmatrix} = \alpha_1 + \alpha_2 \neq 0 \quad \text{identified}$$

The first equation is just identified and the second equation is over identified.

ASSIGNMENT

EC2203 QUANTITATIVE METHODS IN ECONOMICS II

(a) ***Assess the identifiability of the following three-equation model of the goods market by applying the rank and order conditions:***

$$C_t = \alpha_{10} + \alpha_{11} Y_t + \alpha_{12} Y_{t-1} + u_{1t}$$
$$I_t = \beta_{20} + \beta_{21} Y_t + \beta_{22} Y_{t-1} + \beta_{23} I_{t-1} + u_{2t}$$
$$Y_t = C_t + I_t + G_t$$

endogenous variables: C_t, I_t, Y_t

exogenous variable: G_t

(b) ***Assess the identifiability of the following three-equation model by applying the rank and order conditions:***

$$Y_i = \beta_0 + \beta_1 H_i + \beta_2 A_i + \beta_3 E_i + u_{1i}$$
$$H_i = \delta_0 + \delta_1 Y_i + \delta_2 A_i + u_{2i}$$
$$M_i = \gamma_0 + \gamma_1 Y_i + \gamma_2 H_i + u_{3i}$$

endogenous variables: Y_i, H_i, M_i

exogenous variable: A_i, E_i

SOLUTIONS

(a) *Assess the identifiability of the following three-equation model of the goods market by applying the rank and order conditions:*

$$C_t = \alpha_{10} + \alpha_{11} Y_t + \alpha_{12} Y_{t-1} + u_{1t}$$
$$I_t = \beta_{20} + \beta_{21} Y_t + \beta_{22} Y_{t-1} + \beta_{23} I_{t-1} + u_{2t}$$
$$Y_t = C_t + I_t + G_t$$

endogenous variables: C_t, I_t, Y_t

exogenous variable: G_t

In matrix terms:

$$\begin{bmatrix} C_t \\ I_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha_{11} \\ 0 & 0 & \beta_{21} \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_t \\ I_t \\ Y_t \end{bmatrix} + \begin{bmatrix} \alpha_{10} & 0 & \alpha_{12} & 0 \\ \beta_{20} & 0 & \beta_{22} & \beta_{23} \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ G_t \\ Y_{t-1} \\ I_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ 0 \end{bmatrix}$$

We have

$$[(I-B)\,|\,(-G)] = \begin{bmatrix} 1 & 0 & -\alpha_{11} & -\alpha_{10} & 0 & -\alpha_{12} & 0 \\ 0 & 1 & -\beta_{21} & -\beta_{20} & 0 & -\beta_{22} & -\beta_{23} \\ -1 & -1 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

	No. of zeros	Order condition
1st equation:	3	over identified
2nd equation:	2	just identified

Rank condition:

1st equation: the relevant array is $\begin{bmatrix} 1 & 0 & -\beta_{23} \\ -1 & -1 & 0 \end{bmatrix}$

We have eg. $\begin{vmatrix} 1 & -\beta_{23} \\ -1 & 0 \end{vmatrix} = -\beta_{23} \neq 0$, so 1st equation is identified.

2nd equation: $\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = -1 \neq 0$, so 2nd equation is identified.

The first equation is overidentified, and the second equation is just identified.

(b) *Assess the identifiability of the following three-equation model by applying the rank and order conditions:*

$$Y_i = \beta_0 + \beta_1 H_i + \beta_2 A_i + \beta_3 E_i + u_{1i}$$
$$H_i = \delta_0 + \delta_1 Y_i + \delta_2 A_i + u_{2i}$$
$$M_i = \gamma_0 + \gamma_1 Y_i + \gamma_2 H_i + u_{3i}$$

endogenous variables: Y_i, H_i, M_i
exogenous variable: A_i, E_i

In matrix terms:

$$\begin{bmatrix} Y_i \\ H_i \\ M_i \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 & 0 \\ \delta_1 & 0 & 0 \\ \gamma_1 & \gamma_2 & 0 \end{bmatrix} \begin{bmatrix} Y_i \\ H_i \\ M_i \end{bmatrix} + \begin{bmatrix} \beta_0 & \beta_2 & \beta_3 \\ \delta_0 & \delta_2 & 0 \\ \gamma_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ A_i \\ E_i \end{bmatrix} + \begin{bmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{bmatrix}$$

$$[(I-B) \mid (-G)] = \begin{bmatrix} 1 & -\beta_1 & 0 & -\beta_0 & -\beta_2 & -\beta_3 \\ -\delta_1 & 1 & 0 & -\delta_0 & -\delta_2 & 0 \\ -\gamma_1 & -\gamma_2 & 1 & -\gamma_0 & 0 & 0 \end{bmatrix}$$

	No. of zeros	Order condition
1st equation:	1	not identified
2nd equation:	2	just identified
3rd equation:	2	just identified

We apply the rank condition to the second and third equations:

2nd equation: $\begin{vmatrix} 0 & -\beta_3 \\ 1 & 0 \end{vmatrix} = \beta_3 \neq 0$, so 2nd equation identified

3rd equation: $\begin{vmatrix} -\beta_2 & -\beta_3 \\ -\delta_2 & 0 \end{vmatrix} = -\beta_3\delta_2 \neq 0$, so 3rd equation identified

The first equation is not identified. The second and third equations are both just identified.