Lecture 13. Introduction to dynamic models: Part 1

13.1. Introduction

This lecture is intended to cover everything you need to know about dynamic models for the remainder of the QMII course. It comes in two parts, dealing with the following topics:

Part 1 Section 13.2. Dynamic structural equation systems: some basic concepts
Section 13.3. Another example of a dynamic structural equation model
Section 13.4. The partial adjustment hypothesis

Part 2 Section 13.5. Expectations models
Section 13.6. Stochastic dynamic models
Section 13.7. The rational expectations hypothesis

You are expected to work through Part 2 over Reading Week. Part 1 (which you should work through this week) is the most important part in terms of your project work, and the material we will cover after Reading Week.

Time is a crucial factor in many areas of economics. For example, 'old habits die hard', so when circumstances change, firms and households may not be able to adjust their behaviour immediately. It may take some time before it is recognised that circumstances *have* changed! Large capital investment decisions may have long-lasting consequences e.g. building a ship takes years. In many economic decisions (e.g. buying company shares), the conditions prevailing at the moment of decision are less important than those expected to prevail at some future times.

In this lecture, we consider econometric models designed to describe the behaviour of economic variables over time. We shall thus be focusing on models for time series data. In Lecture 11, we used the following consumption function equation to introduce some basic concepts:

$$C_t = \alpha + \beta Y_t + u_t$$

We now regard this as a <u>static</u> equation. The only determinant of <u>current</u> consumption is <u>current</u> income. If Y_t increases by one unit in period t, C_t is assumed to increase by β units <u>in the same period</u>. In other words, all 'adjustments' are assumed to be completed in a single period.

An example of a <u>dynamic</u> consumption function equation is obtained if it is assumed that there is a one-period delay before consumption responds to income. We can write

$$C_t = \alpha + \beta Y_{t-1} + u_t$$

and say that consumption depends on income 'with a one period lag'. Another example arises if it is assumed that C_t depends on current income, and also on consumption last period ('lagged consumption') e.g. because consumer's habits take time to change. We could then write

$$C_t = \alpha + \beta Y_t + \gamma C_{t-1} + u_t$$

This equation is an example of a <u>difference equation</u>. The term 'difference equation' applies generally to a relation among the values of a variable at various points in time (Chiang, *Fundamental Methods of Mathematical Economics*, provides a superb introductory treatment of difference equations. I strongly recommend that you read him).

13.2. Dynamic structural equation systems: some basic concepts

In this section, we look at systems of <u>structural equations</u> in which some of the equations contain <u>dynamic</u> elements as discussed briefly in the introduction. ***Such dynamic systems describe the behaviour over time of the endogenous variables, and we shall be interested in studying the nature of the <u>time path</u> of each variable, as well as the question of whether time paths converge to <u>equilibrium positions</u>.*** As usual, we shall develop the principal ideas in the context of a simple Keynesian national income model. As the disturbance term is not relevant to the discussion in this section, we can neglect it for the moment, and focus on the systematic parts of the model.

Let us amend the two-equation national income model of Lecture 11 by incorporating into it a one-period delay in the consumption function:

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

Suppose investment is equal to the constant I in every period ie. $I_t = I$ for t = 1, 2, 3, . . We now ask: what is the nature of the equilibrium position?

The equilibrium values of the endogenous variables are defined as the values that, once achieved, are maintained thereafter. The equilibrium values of consumption and income, denoted by C^e and Y^e can be obtained by solving the following pair of equations:

$$C^e = \alpha + \beta Y^e$$
$$Y^e = C^e + I$$

Notice that the time subscripts have been eliminated from the model: by the definition of equilibrium, consumption and income do not vary over time once equilibrium has been attained. Substituting the second equation into the first, and rearranging gives

$$C^e = \frac{\alpha + \beta I}{1-\beta}$$

Substituting this result into the second equation and collecting terms gives

$$Y^e = \frac{\alpha + I}{1-\beta}$$

Thus, the equilibrium position (or the 'solution') of the dynamic structural equation system

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

is $C^e = \frac{\alpha + \beta I}{1-\beta}$ and $Y^e = \frac{\alpha + I}{1-\beta}$. Note that this solution is exactly the same as for the static model discussed in Lecture 11 (see page 5 of the handout for Lecture 11). This is because

$$C^e = \alpha + \beta Y^e$$
$$Y^e = C^e + I$$

is a static model. ***The equilibrium position of the dynamic system has thus been found as the solution of a static model obtained from the dynamic system by ignoring differences in the time subscripts.*** This statement generalises: ***For every dynamic system having an equilibrium position, there is a corresponding static system which describes that position.*** However, one static system may describe the equilibrium position of a number of different dynamic systems, which all 'collapse' to the same static model when time subscripts are eliminated. For example,

$$C_t = \alpha + \beta Y_{t-2}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

is a different dynamic system to the one introduced earlier because it involves a two-period lag, but it has the same equilibrium solution.

Now we consider the time path of the variables and, in particular, whether this equilibrium position can be reached. The starting point of the path is denoted by C_0 and Y_0, called the initial conditions or the initial values. ***A dynamic structural equation system gives the time path of C_t and Y_t in terms of initial conditions, parameter values, and values of exogenous variables. If these paths tend to equilibrium irrespective of the initial conditions, the system is said to be stable.***

It is convenient at this point to carefully distinguish the income variable in the two forms in which it appears in the structural equation system: we call Y_{t-1} a lagged endogenous variable, and Y_t (and also C_t) current endogenous variables. The two equations of the model then describe the determination of the two current endogenous variables in terms of lagged endogenous and exogenous variables, and these together are called predetermined variables. ***The structural equation system has a reduced form, just as in the static case, but the previous definition of a reduced form equation requires a slight extension in the dynamic case: it describes the determination of a current endogenous variable in terms of parameters and predetermined variables.*** Note that, with this definition, the consumption function in our simple model

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

is already a reduced form equation. Substituting the consumption function into the income equation and rearranging yields the following complete reduced form:

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = \alpha + \beta Y_{t-1} + I_t$$

The reduced form allows us to calculate the time paths of C_t and Y_t step-by-step. Given an initial condition Y_0 and the value I_1, the reduced form above gives C_1 and Y_1. Then given Y_1 and I_2, the reduced form above gives C_2 and Y_2. Given Y_2 and I_3, the reduced form gives us C_3 and Y_3 etc..

The reduced form may not be the most convenient solution form if we wish to describe the evolution of a single endogenous variable. In the example above, it is necessary to solve both reduced form equations period by period in order to obtain the path of counsumption, for instance. If our model consisted of hundreds of equations, but we were only interested in the time path of one endogenous variable, it would be extremely inefficient to attempt to obtain this time path by iterating through all the reduced form equations. ***We therefore introduce the concept of a final equation, which expresses a current endogenous variable in terms of exogenous variables and lagged values of itself, but no other endogenous variable.*** With this definition, the reduced form equation for income above is already a final equation: it describes the evolution of income without reference to the behaviour of consumption. ***However, in the consumption reduced form equation we need to get rid of Y_{t-1} to obtain a final equation.*** We can do this by using the identity

$$Y_{t-1} = C_{t-1} + I_{t-1}$$

Substituting this into our dynamic consumption function above we get the following final equations for our model:

$$C_t = \alpha + \beta C_{t-1} + \beta I_{t-1}$$
$$Y_t = \alpha + \beta Y_{t-1} + I_t$$

The advantage of these final equations is that they can be used independently of one another to describe the time path of the respective endogenous variables. Each is a difference equation in a single endogenous variable.

A final equation allows the calculation of the time path of an endogenous variable period-by-period (without reference to the other endogenous variables in the model). In other words, you you can get C_t for example, then C_{t+1}, then C_{t+2}, and so on. ***However, for some purposes, it may be necessary to have a general expression for the value of the endogenous variable at any point in time as a function of time, exogenous variables, initial conditions and structural parameters.*** The solution of the difference equation provides such an expression, and we now briefly look at the solution of first-order difference equations in a general context (for an excellent and comprehensive introductory treatment of difference equations, see Chiang, Chapters 16, 17 and 18). You should be able to grasp the following discussion even if you have not studied difference equations before. The ***linear first-order non-homogeneous difference equation with coefficients a and b*** is

$$y_t = a + by_{t-1}$$

Let y^e denote the equilibrium value of y i.e. the value such that if y is set at y^e, then it will remain there for ever. Then in equilibrium, we must have

$$y^e = a + by^e$$

which implies that

$$y^e = \frac{a}{1-b}$$

or

$$a = (1 - b)y^e$$

Substituting this expression for a into the original difference equation above we get

$$y_t = (1 - b)y^e + by_{t-1}$$

Rearranging this gives us a 'homogeneous' difference equation of the following form:

$$y_t - y^e = b(y_{t-1} - y^e)$$

Given an initial condition y_0, we can then write

$$y_1 - y^e = b(y_0 - y^e)$$

Using this expression, we can then move forward one period and write

$$y_2 - y^e = b(y_1 - y^e) = b\{b(y_0 - y^e)\} = b^2(y_0 - y^e)$$

Using this expression in turn, we can move forward again by one period to get

$$y_3 - y^e = b(y_2 - y^e) = b\{b^2(y_0 - y^e)\} = b^3(y_0 - y^e)$$

and so on. We can write the general expression for time t as

$$y_t - y^e = b^t(y_0 - y^e)$$

This is called the general solution of the difference equation, which allows us to get the value of y_t for any time t. We assume that the system did not actually start out in equilibrium, so that $y_0 \neq y^e$ (otherwise there would be no point in studying whether the system converges to equilibrium!). Then the term on the right hand side of the general solution of the difference equation $(y_0 - y^e)$ does not equal zero. ***Since y_t approaches y^e if and only if the right hand side of the general solution goes to zero as $t \to \infty$, we need $b^t \to 0$ as $t \to \infty$.*** The necessary and sufficient condition for $b^t \to 0$ as $t \to \infty$ is $|b| < 1$ (i.e. $-1 < b < 1$), which is called the stability condition.

Applying this general result to the final equations of our dynamic model (which are difference equations, remember) we see that ***in the first-order case, whether a dynamic system converges to an equilibrium depends on the magnitude of the coefficient of the lagged dependent variable in the final equation.*** If $|b| < 1$, the values of y_t approach the equilibrium value y^e. If $|b| > 1$, the system is said to 'diverge' or 'explode'.

Given that $|b| < 1$, convergence to equilibrium will be oscillatory if $b < 0$, and smooth if $b > 0$. You get oscillations when $b < 0$ because b^t is then positive when t is even, and negative when t is odd. So b^t alternates in sign for t = 1, 2, 3, 4, 5, 6, . . . when $b < 0$.

To apply these insights, let us return to our structural equation model:

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

We saw that the final equation for consumption is

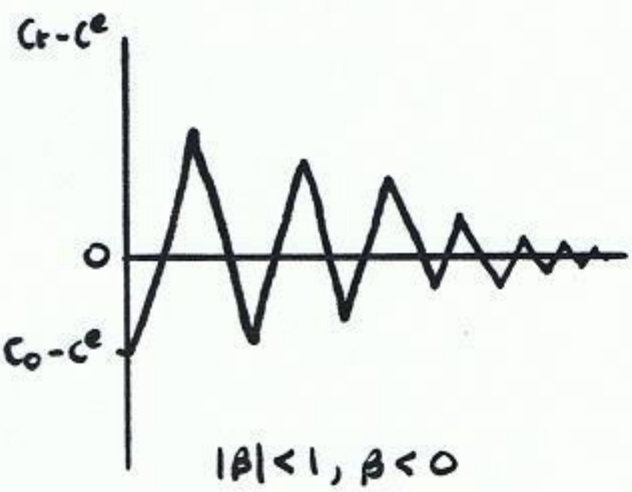
$$C_t = \alpha + \beta C_{t-1} + \beta I_{t-1}$$

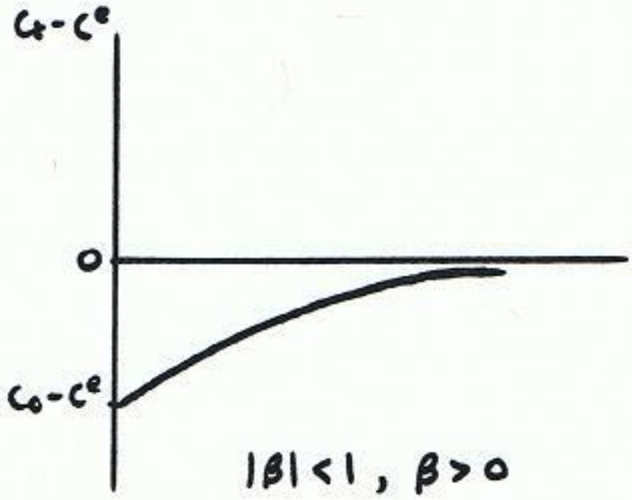
Assuming that $I_t = I$, we had obtained the equilibrium position as $C^e = \dfrac{\alpha + \beta I}{1-\beta}$. This is analogous to our little difference equation example above: $y^e = \dfrac{a}{1-b}$. In the same way that we got the general solution for that difference equation, we can also get the following general solution for C_t in our dynamic model:

$$C_t - C^e = \beta^t(C_0 - C^e)$$

The stability condition is $|\beta| < 1$, and we expect to get a stable solution since β is the marginal propensity to consume. This would have to be tested by estimating the dynamic structural equation model using time series data.

The two types of convergent solution are sketched in the diagrams below, assuming that $C_0 < C^e$.





Note that the assumption that I_t is constant implies a fixed equilibrium value C^e. If we were to specify that I_t varied over time in some way, following an upward linear trend, say, then the equilibrium position would be a moving one, and the path of C_t would approach this equilibrium path rather than the horizontal line in the diagrams above.

An important point to make is that exactly the same stability condition for our dynamic structural equation model (i.e. $|\beta| < 1$) is obtained by examining the final equation for income

$$Y_t = \alpha + \beta Y_{t-1} + I_t$$

(check this for yourself). In other words, it does not matter whether we look at the final equation for C_t or Y_t. ***This is because both final equations have the same autoregressive structure - the number of lagged values of the dependent variable that appear on the right hand side, and the coefficients with which they appear, are the same.*** This is a perfectly general result. ***Even if you have a dynamic structural equation model involving hundreds of equations, you only have to look at one final equation in order to work out the stability condition for the entire model.*** In practice, we examine whichever final equation is easiest to derive.

13.3. Another example of a dynamic structural equation model

Although we only need one final equation to work out the stability condition for an entire model, it sometimes happens that other final equations are also required for other purposes. In this case, the knowledge that they must all have the same autoregressive structure can considerably reduce the computational difficulties involved in deriving them. To illustrate this, we consider a three-equation dynamic model based on the following static model:

$$C = \alpha + \beta Y$$
$$Y = C + I$$
$$I = \gamma Y + A$$

endogenous variables: C, Y, I
exogenous variables: A

(recall that a static model can serve as a description of the equilibrium position for one or more dynamic models. It is useful to begin the specification of a dynamic model by considering first the static model that describes its equilibrium (if it has one)). In this model, investment now has a component that depends on income, together with an 'autonomous' component A. The dynamic model we shall use retains the lagged consumption function of the previous section, and specifies that the other two equations involve only current dated variables. Thus, the dynamic model we shall study in this section is as follows:

$$C_t = \alpha + \beta Y_{t-1}$$
$$Y_t = C_t + I_t$$
$$I_t = \gamma Y_t + A$$

endogenous variables: C_t, Y_t, I_t
exogenous variables: A

It is simpler to assume that the exogenous variable has a constant value A, say, so that there is a fixed equilibrium position given as a function of A, rather than a moving equilibrium position which is a function of A_t.

Substituting the equations for C_t and I_t into the income identity gives

$$Y_t = \alpha + \beta Y_{t-1} + \gamma Y_t + A$$

Solving this equation for Y_t gives the reduced form equation for income:

$$Y_t = \frac{\alpha + A}{1-\gamma} + \frac{\beta}{1-\gamma} Y_{t-1}$$

Notice that this equation is already a final equation: it contains no lagged endogenous variables other than lagged values of Y_t, the dependent variable. The first term on the right hand side corresponds to the constant 'a' of the non-homogeneous difference equation we looked at earlier: $y_t = a + by_{t-1}$. The coefficient of Y_{t-1} corresponds to the coefficient b, required to be less than 1 in absolute value for stability. A general solution for Y_t in terms of an initial condition, an equilibrium value (itself a function of α, β, γ and A) and time can then be easily derived (you could do this for yourself as an exercise). For our present purposes, we only need the stability condition, which by inspection is

$$-1 < \frac{\beta}{1-\gamma} < 1$$

If this condition is satisfied, then the entire three-equation system must be stable. It converges to the equilibrium position calculated as the solution of the static model.

The value of $\frac{\beta}{1-\gamma}$ determines the stability properties of all three endogenous variables, as stated above, but if the final equations for consumption or investment are required for other purposes, ***then knowledge of the autoregressive structure of the final equation for income is useful.*** For example, the consumption function has no current endogenous variable on the right hand side, and so is already a reduced form equation. ***To obtain the final equation, we need to express the right hand side in terms of the exogenous variable and lagged values of consumption, not income or investment.*** We cannot now substitute from the lagged income identity (as we did in the last section) because this introduces the lagged value of another endogenous variable (I_t). ***However, subtracting $\frac{\beta}{1-\gamma}$ times the lagged consumption equation produces an expression in Y_{t-1} and Y_{t-2} which can be simplified by substituting from the final equation for income above.*** This does the trick:

$$C_t - \frac{\beta}{1-\gamma} C_{t-1} = \alpha + \beta Y_{t-1} - \frac{\beta}{1-\gamma} (\alpha + \beta Y_{t-2})$$

$$= \alpha(1 - \frac{\beta}{1-\gamma}) + \beta(Y_{t-1} - \frac{\beta}{1-\gamma} Y_{t-2})$$

$$= \alpha(1 - \frac{\beta}{1-\gamma}) + \beta(\frac{\alpha + A}{1-\gamma})$$

The second term on the right hand side of the last equals sign comes from the final equation for income obtained earlier. So the final equation for consumption is

$$C_t = \alpha(1 - \frac{\beta}{1-\gamma}) + \beta(\frac{\alpha + A}{1-\gamma}) + \frac{\beta}{1-\gamma} C_{t-1}$$

As an exercise (in elementary algebra!) you should verify for yourself that this last equation can be simplified to

$$C_t = (\alpha + \frac{\beta A}{1-\gamma}) + \frac{\beta}{1-\gamma} C_{t-1}$$

So notice, then, that we have used the autoregressive structure of the final equation for income to simplify the derivation of the final equation for consumption. Again, the bracketed expression corresponds to 'a' in the simple difference equation we looked at earlier (i.e. $y_t = a + by_{t-1}$), and $\frac{\beta}{1-\gamma}$ corresponds to the parameter b. Substituting these values of a and b into the formula a/(1-b) gives the static equilibrium level of consumption as a function of α, β, γ and A.

13.4. The partial adjustment hypothesis

So far, we have looked at dynamic structural equation systems, but we have not really looked in detail at how dynamic terms can arise in equations. In this section, we study the famous partial adjustment hypothesis. Again, we shall neglect the disturbance term for the moment.

We suppose that there is a 'desired' or 'planned' level of some variable that an economic agent would like to achieve at time t, and denote this as y_t^*. For example, this might represent a firm's optimal capital stock. ***But assume that there are frictions, delays, costs of doing business, habit persistence, etc.. such that the desired level cannot be entirely achieved in a single period.***

Starting from the previously existing level y_{t-1}, the change required to attain the desired level is

$$(y_t^* - y_{t-1})$$

but we assume that the actual change $(y_t - y_{t-1})$ is only a fraction of this. Assuming that the proportion achieved is $(1-\gamma)$, where $0 < \gamma < 1$, the partial adjustment hypothesis can be written

$$y_t - y_{t-1} = (1-\gamma)(y_t^* - y_{t-1})$$

or equivalently,

$$y_t = (1-\gamma)y_t^* + \gamma y_{t-1}$$

Small values of γ imply relatively quick adjustment, and if $\gamma = 0$ adjustment is complete, not partial, in a single period. Larger values of γ imply that the past value of the variable exerts a greater influence, and if $\gamma = 1$, nothing ever changes.

We need to say a bit more about the desired value y_t^*. ***If data on this are available, then the above equation can be estimated.*** But this is very rare, and we must usually provide a theoretical explanation of how the unobservable y_t^* is determined by observable variables. For example, a firm's (unobservable) optimal capital stock might depend on the prices of its output, and the factors of production which can be observed.

As a simple illustration, suppose that the desired level depends on a single explanatory variable x_t as follows:

$$y_t^* = \alpha + \beta x_t$$

Then on substituting this expression for y_t^* in the equation $y_t = (1-\gamma)y_t^* + \gamma y_{t-1}$ we get the following relation between observable variables:

$$y_t = \alpha(1-\gamma) + \beta(1-\gamma)x_t + \gamma y_{t-1}$$

It is used to be said that the partial adjustment hypothesis is 'ad hoc' (i.e. not derived from a fully worked theoretical model of economic behaviour), but it has been shown that a justification can be provided in terms of a cost minimisation procedure. Suppose that a firm selects the value of the variable y_t so as to minimise the weighted sum of 'disequilibrium' and 'adjustment' costs:

$$C_t = a_1(y_t - y_t^*)^2 + a_2(y_t - y_{t-1})^2$$

The first term on the right hand side represents the costs incurred by being away from the optimum or desired position y_t^*, and the second term represents the costs of changing y, for example 'hiring and firing' costs if y represents the size of the firm's labour force.

The above cost function is quadratic, and implies that positive and negative deviations are equally expensive. To find the value of y_t that minimises C_t, we equate the first partial derivative to zero:

$$\partial C_t/\partial y_t = 2a_1(y_t - y_t^*) + 2a_2(y_t - y_{t-1}) = 0$$

So we have $2(a_1 + a_2)y_t - 2a_1y_t^* - 2a_2y_{t-1} = 0$. Solving this for y_t gives

$$y_t = [a_1/(a_1+a_2)]y_t^* + [a_2/(a_1+a_2)]y_{t-1}$$

(verify this for yourself; it's easy!). On setting $\gamma = a_2/(a_1+a_2)$ so that $1-\gamma = a_1/(a_1+a_2)$ we get

$$y_t = (1-\gamma)y_t^* + \gamma y_{t-1}$$

which is the partial adjustment hypothesis! Subtracting y_{t-1} from both sides we can write it in more familiar form

$$y_t - y_{t-1} = (1-\gamma)(y_t^* - y_{t-1})$$

If adjustment costs are relatively important, the coefficient a_2 will be relatively large, which implies a value of γ near to 1, and relatively slow adjustment. Obviously, there are no costs of adjustment if $a_2 = 0$, and the firm moves immediately to the desired position y_t^*.

(End of Lecture 13 Part 1)

Lecture 13. Introduction to dynamic models: Part 2

13.5. Expectations models

This is the second part of Lecture 13, which deals with dynamic models in econometrics. The following topics are covered in the two parts:

Part 1 Section 13.2. Dynamic structural equation systems: some basic concepts
Section 13.3. Another example of a dynamic structural equation model
Section 13.4. The partial adjustment hypothesis

Part 2 Section 13.5. Expectations models
Section 13.6. Stochastic dynamic models
Section 13.7. The rational expectations hypothesis

It is convenient to briefly review the material of Part 1. It introduced you to the 'workings' of dynamic structural equation models. These describe the behaviour of endogenous variables over time. We are usually interested in the time paths of these variables, and the conditions under which they tend to an equilibrium value. We extended the concept of 'reduced form' equations by saying that, in a dynamic context, these explain endogenous variables in terms of exogenous variables, 'lagged' endogenous variables and parameters. ***Exogenous variables and 'lagged' endogenous variables are classified together as predetermined variables.*** Thus, we say that a dynamic structural equation system has a reduced form in which the endogenous variables of the model are explained in terms of predetermined variables and parameters only. We also introduced the concept of a 'final' equation which expresses an endogenous variable in terms of exogenous variables, lagged values of itself, and parameters; no other endogenous variables or lagged endogenous variables appear in a final equation. The final equation for any one of the endogenous variables of a dynamic structural equation system can be used to determine the stability condition for the entire system. This is a condition that must be satisfied by certain parameters of the model if the time paths of the endogenous variables are to converge to equilibrium positions. Finally, we considered the partial adjustment hypothesis as one way of explaining how 'dynamic' terms can arise in equations.

A second group of models that can give rise to dynamic equations involves expectations or anticipations of the future values of a variable. ***It is very important here to distinguish between two possible scenarios: the first arises when the variable about which expectations are formed is an exogenous variable; the second case is when expectations are being formed with regard to an endogenous variable.*** The most famous model of expectations with regard to an exogenous variable is the adaptive expectations model, which we shall discuss in this section. The most famous approach to modelling expectations with regard to an endogenous variable is the rational expectations hypothesis, which we shall discuss in Section 13.7.

In this section, we assume that behaviour with respect to a particular dependent variable is influenced by expected or anticipated values of an exogenous explanatory variable. The expectations might relate to such variables as sales, prices, incomes, or interest rates depending on the particular problem at hand. For example, a retailer's stock of goods might depend on the sales he/she expects to make in the next period, money balances may react to the level of income expected in the next period, raw material stocks may be adjusted to reflect expected future prices, etc.. The discussion in this (and the next) section is based on the simple behavioural relation

$$y_t = \alpha + \beta \bar{x}_{t+1}$$

where $\bar{x}_{t+1}$ denotes the forecast or expected value of x_{t+1}, formed at time t. For example, y_t might denote current inventories, and $\bar{x}_{t+1}$ might denote forecast sales. Although actual observations on the expectational variable might be available from, say, surveys on sales anticipations or business investment intentions, these instances are relatively rare, and in most cases anticipations are unobservable. ***To obtain a model in terms of observable variables, it is necessary to add an assumption about the formation of expectations.***

A simple assumption is that currently observed conditions are expected to prevail in the next period, giving what are known as 'naive' or 'no-change' expectations:

$$\bar{x}_{t+1} = x_t$$

If the assumption of naive expectations is incorporated in the above behavioural relation we get $y_t = \alpha + \beta x_t$ which is indistinguishable from a simple static relation between the observed values of y and x. A slightly less simplistic assumption is the so-called 'same-change' forecasting rule, in which the next period's value is anticipated to differ from the current value by the same amount that the current value has been observed to differ from the previous value:

$$\bar{x}_{t+1} - x_t = x_t - x_{t-1}$$

More generally, it might be assumed that economic agents use their knowledge of the behaviour of the variable over time to calculate forecasts based on past data. ***If successive x-values are correlated with one another, this autocorrelation indicates the extent to which current and past values are helpful in forecasting future values.*** To be more precise, we assume that economic agents have information about the data generating process (DGP) of the x-variable, and that they use this information optimally. ***A statistical model of the x-variable is constructed, and then the forecast calculated.***

A basic building block of such models is a 'white noise' process, involving 'purely random' variables conventionally denoted by ε_t. ***A white noise process has three defining characteristics:***
(1). $E[\varepsilon_t] = 0$ for all t;
(2). $E[\varepsilon_t^2] = \sigma^2$ for all t i.e. the variance of ε_t is constant over time;
(3). $E[\varepsilon_t \varepsilon_s]=0$ for $t \neq s$ i.e. there is zero covariance between values of ε_t at different time points.

The fact that there is no correlation between any pair of values implies that current and past data contain no information that is useful in forecasting the next value, so the best estimate is simply the mean value i.e. zero. ***Of course, economic time series typically have non-zero autocorrelation (e.g. prices usually increase steadily over time, so high current values lead to even higher future values) but the non-autocorrelated nature of ε_t makes it convenient to work with when constructing more general models.***

Example 1: Suppose we happen to know that x_t obeys the *first-order autoregression*

$$x_t = \rho x_{t-1} + \varepsilon_t \qquad |\rho| < 1$$

This has the form of a regression of x_t on its own past value, hence the term 'autoregression', and it is of 'first order' because only a one-period lagged value of x appears on the right hand side. We know that the value we seek to forecast will also be generated in the same way:

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}$$

and since we have no information at time t that is useful in forecasting ε_{t+1} apart from its zero mean value, but x_t is known, a sensible forecast for us to make is

$$\bar{x}_{t+1} = \rho x_t$$

This implies that once we have observed the current value x_t, then previous data $x_{t-1}, x_{t-2}, \ldots$, convey no additional information that would help us in forecasting x_{t+1}.

We said that the forecast $\bar{x}_{t+1} = \rho x_t$ is the 'sensible' one to make, given that x_t follows a first-order autoregressive process, but it is also the 'optimal' forecast in a very important sense: ***it minimises the mean squared error of the forecast***. To understand this, note that the forecast error is

$$x_{t+1} - \bar{x}_{t+1}$$

and we wish to make the expected value of the square of this as small as possible i.e. we wish to minimise

$$E[(x_{t+1} - \bar{x}_{t+1})^2]$$

(We wish to minimise the expected square of the forecast error because we want to 'penalise' negative deviations just as much as positive ones). Let us define an 'arbitrary' forecasting rule in terms of current and past data:

$$\bar{x}_{t+1} = a_0 x_t + a_1 x_{t-1} + a_2 x_{t-2} + \cdots = \sum_{j=0}^{\infty} a_j x_{t-j}$$

Notice that this equals our 'sensible' forecasting rule $\bar{x}_{t+1} = \rho x_t$ only when $a_0 = \rho$, and $a_1 = a_2 = \cdots = 0$. Using the arbitrary forecasting rule, and assuming that x_t follows a first-order autoregressive process (so that we have $x_{t+1} = \rho x_t + \varepsilon_{t+1}$) we get the following expression for the mean squared error of the forecast:

$$E[(x_{t+1} - \bar{x}_{t+1})^2] = E[(\rho x_t + \varepsilon_{t+1} - \sum_{j=0}^{\infty} a_j x_{t-j})^2]$$

$$= E[\varepsilon_{t+1}^2] + E[(\rho x_t - \sum_{j=0}^{\infty} a_j x_{t-j})^2] + \textit{expected value of cross-product between } \varepsilon_{t+1} \textit{ and terms in } x_t, x_{t-1}, x_{t-2}, \ldots$$

The expected value of the cross-product between ε_{t+1} and terms in $x_t, x_{t-1}, x_{t-2}, \ldots$ is zero, because ε_{t+1} is uncorrelated with any previous data. Thus we end up with the following expression for the mean squared error of the forecast:

$$E[(x_{t+1} - \bar{x}_{t+1})^2] = E[\varepsilon_{t+1}^2] + E[(\rho x_t - \sum_{j=0}^{\infty} a_j x_{t-j})^2]$$

Now, we cannot do anything to minimise the size of the term $E[\varepsilon_{t+1}^2]$, but we *can* make the term $E[(\rho x_t - \sum_{j=0}^{\infty} a_j x_{t-j})^2]$ equal to zero (its minimum value) by setting $a_0 = \rho$, and $a_1 = a_2 = \cdots = 0$. ***Thus, we can make the mean squared error of the forecast as small as possible by using our 'sensible' forecasting rule*** $\bar{x}_{t+1} = \rho x_t$, ***and it is in this sense that our 'sensible' rule is also the 'optimal' one when*** x_t ***follows a first-order autoregressive process.***

To make sure that we do not lose sight of what we are doing, let us briefly recap at this point. We are assuming in this section that a dependent variable y_t is affected by the expected future value of an exogenous variable $\bar{x}_{t+1}$ according to the following simple behavioural relation:

$$y_t = \alpha + \beta \bar{x}_{t+1}$$

The expectation term on the right hand side is unobservable, so in order to get an equation in terms of observable variables, we are using statistical models of the behaviour of the variable x_t over time to calculate $\bar{x}_{t+1}$ 'optimally'. We have just demonstrated that when x_t follows the statistical model known as the 'first-order autoregressive process', the 'optimal' forecast of x_{t+1} is $\bar{x}_{t+1} = \rho x_t$.

Now let us continue. An interesting point to make is that as $\rho \to 1$, the autoregressive model becomes

$$x_t = x_{t-1} + \varepsilon_t$$

or equivalently

$$x_t - x_{t-1} = \varepsilon_t$$

or using the 'difference operator' Δ,

$$\Delta x_t = \varepsilon_t$$

This is the very famous 'random walk' model, in which the first difference of the x-series is purely random. Random walk models have been extensively applied to share prices in studies of stock exchange behaviour. ***The optimal forecasting rule is now*** $\bar{x}_{t+1} = x_t$, ***and so we see that the 'naive' forecast is actually the 'best' forecast if the series being predicted follows a random walk.***

Example 2: We now modify the first-order autoregression model by making its error term a 'moving average' of two successive random shocks:

$$x_t = \rho x_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1} \qquad |\theta| < 1$$

This is the famous autoregressive-moving average model, shortened to 'ARMA' in the literature. You will become very familiar with this model (and its extensions) if you go on to do a Master's degree. It says that a given random shock has an effect that persists for two periods, since it affects the contemporaneous x-value, and also the following value.

Given that the exogenous variable x_t follows an ARMA process, it can easily be demonstrated that the 'optimal' forecast of x_{t+1} (i.e. the one that minimises the mean squared error of forecast) is

$$\begin{aligned}\bar{x}_{t+1} &= (\rho - \theta)\sum_{j=0}^{\infty} \theta^j x_{t-j} \\ &= (\rho - \theta)(x_t + \theta x_{t-1} + \theta^2 x_{t-2} + \theta^3 x_{t-3} + \cdots) \\ &= (\rho - \theta)x_t + (\rho - \theta)(\theta x_{t-1} + \theta^2 x_{t-2} + \theta^3 x_{t-3} + \cdots)\end{aligned}$$

We can get a neat expression for this by noting that the second term on the right hand side of the last 'equals sign' is just $\theta\bar{x}_t$, where $\bar{x}_t$ is the forecast of x_t calculated at time t-1 using the same forecasting rule. Thus, we get the following expression for the optimal forecasting rule for x_{t+1} when x_t follows an ARMA process:

$$\bar{x}_{t+1} = (\rho - \theta)x_t + \theta\bar{x}_t$$

If we set $\rho = 1$ in this equation, we get the very famous adaptive expectations model:

$$\bar{x}_{t+1} = (1 - \theta)x_t + \theta\bar{x}_t$$

We can rewrite this in a different way by adding and subtracting $\bar{x}_t$ on the right hand side. We get

$$\begin{aligned}\bar{x}_{t+1} &= \bar{x}_t + (1 - \theta)x_t + \theta\bar{x}_t - \bar{x}_t \\ &= \bar{x}_t + (1 - \theta)x_t + (\theta - 1)\bar{x}_t \\ &= \bar{x}_t + (1 - \theta)x_t - (1 - \theta)\bar{x}_t \\ &= \bar{x}_t + (1 - \theta)(x_t - \bar{x}_t)\end{aligned}$$

So we have the following two equivalent ways of writing the adaptive expectations model:

$$\bar{x}_{t+1} = (1 - \theta)x_t + \theta\bar{x}_t$$
$$\bar{x}_{t+1} = \bar{x}_t + (1 - \theta)(x_t - \bar{x}_t)$$

The adaptive expectations model simply says that expectations are amended or 'adapted' in proportion to past forecasting errors, the next period's forecast $\bar{x}_{t+1}$ ***being given by the***

forecast of the current level, $\bar{x}_t$***, amended by a proportion of the current forecast error,*** $(x_t - \bar{x}_t)$. If the parameter θ is close to zero, expectations adapt rapidly to recent data, while if θ is near 1, expectations are slow to change.

Like the partial adjustment hypothesis, the adaptive expectations hypothesis has often been criticised for its apparent 'ad hoc' nature. But the argument above shows that the hypothesis is 'sensible' if we believe that the exogenous variable is generated according to

$$\Delta x_t = \varepsilon_t - \theta\varepsilon_{t-1}$$

(which is just an ARMA process with $\rho = 1$), and that economic agents make 'optimal' forecasts within this framework.

Returning to our original behavioural equation

$$y_t = \alpha + \beta \bar{x}_{t+1}$$

recall that we are exploring ways of getting rid of the unobservable expectations variable on the right hand side. One way of doing this (which you should learn like a 'parrot', because it is a frequently employed technique in many different contexts) is to use the adaptive expectations equation

$$\bar{x}_{t+1} = (1 - \theta)x_t + \theta \bar{x}_t$$

We have

$$y_t - \theta y_{t-1} = \alpha + \beta \bar{x}_{t+1} - \theta(\alpha + \beta \bar{x}_t)$$
$$= \alpha(1 - \theta) + \beta(\bar{x}_{t+1} - \theta \bar{x}_t)$$

We can get rid of the second term on the right hand side by noting that

$$\bar{x}_{t+1} - \theta \bar{x}_t = (1 - \theta)x_t$$

(this is straight from the adaptive expectations equation above). So we can write

$$y_t - \theta y_{t-1} = \alpha(1 - \theta) + \beta(1 - \theta)x_t$$

or equivalently

$$y_t = \alpha(1 - \theta) + \beta(1 - \theta)x_t + \theta y_{t-1}$$

This is the 'final equation' for the adaptive expectations hypothesis. ***An important point is that this is precisely the same final equation as that obtained from the partial adjustment hypothesis (see page 9 of Lecture 13 Part 1), showing that there exists a 'duality' between the adaptive expectations and partial adjustment hypotheses.*** Thus, if we find that a regression equation of this form provides a satisfactory explanation of y_t, it is open to argument whether lags in adjustment or expectational variables are at work. ***Simple regression analysis cannot be used to distinguish between the two hypotheses.*** However, we *can* distinguish between the two if we introduce an error term into the models. This is discussed in the next section.

13.6. Stochastic dynamic models

In the introduction to these lecture notes (Section 13.1) we wrote down a dynamic consumption function that included a random disturbance term, but in the subsequent sections this term has been neglected in order to focus on other aspects of dynamic models. We now consider the consequences of reintroducing the disturbance term. The reasons for the inclusion of a random disturbance term in a dynamic behavioural relation are exactly the same as those discussed in a static context in Lecture 11. ***The random error represents the effect on the dependent variable of a host of omitted influences that are unobserved or unidentified.***

When deriving the final equation for the partial adjustment model (see page 9 of Lecture 13 Part 1) we assumed that the desired level of the variable y_t, denoted by y_t^*, depends on a single explanatory variable x_t as follows:

$$y_t^* = \alpha + \beta x_t$$

Then on substituting this expression for y_t^* in the partial adjustment equation $y_t = (1-\gamma)y_t^* + \gamma y_{t-1}$ we got the following relation between <u>observable</u> variables:

$$y_t = \alpha(1-\gamma) + \beta(1-\gamma)x_t + \gamma y_{t-1}$$

Suppose we now add a random error u_t to the equation for y_t^* to get

$$y_t^* = \alpha + \beta x_t + u_t$$

As usual, we assume that $E[u_t] = 0$. Substituting this stochastic equation into the partial adjustment equation $y_t = (1-\gamma)y_t^* + \gamma y_{t-1}$ we get the following relation between observable variables, which now includes a random disturbance term:

$$y_t = \alpha(1-\gamma) + \beta(1-\gamma)x_t + \gamma y_{t-1} + (1-\gamma)u_t$$

As in static models, the disturbance term $(1-\gamma)u_t$ is interpreted as the <u>non-systematic</u> part of the relation. Notice that it has zero mean: $E[(1-\gamma)u_t] = (1-\gamma)E[u_t] = 0$. The systematic part of this equation describes the determination of the current endogenous variable y_t by the exogenous variable x_t and the lagged endogenous variable y_{t-1} i.e. by the <u>predetermined</u> variables (using the terminology introduced earlier).

We shall now show how the addition of random disturbances makes the partial adjustment and adaptive expectations models <u>observationally distinguishable</u>. ***This is because the error term in the final equation of the partial adjustment model is simply $(1-\gamma)$ times the disturbance term introduced into the equation for y_t^*, whereas the error term in the final equation of the adaptive expectations model is a moving average of the values of the disturbance term at different points in time.*** To see this, let us add an error term to the behavioural relation in the expectations model:

$$y_t = \alpha + \beta \bar{x}_{t+1} + u_t$$

Assuming that the unobservable $\bar{x}_{t+1}$ obeys the adaptive expectations model

$$\bar{x}_{t+1} - \theta \bar{x}_t = (1 - \theta)x_t$$

we obtain a relation between observable variables by transforming in a manner equivalent to that used earlier in this lecture (this is the method I told you to learn like a parrot!):

$$\begin{aligned} y_t - \theta y_{t-1} &= \alpha + \beta \bar{x}_{t+1} + u_t - \theta(\alpha + \beta \bar{x}_t + u_{t-1}) \\ &= \alpha(1 - \theta) + \beta(\bar{x}_{t+1} - \theta \bar{x}_t) + (u_t - \theta u_{t-1}) \\ &= \alpha(1 - \theta) + \beta(1 - \theta)x_t + (u_t - \theta u_{t-1}) \end{aligned}$$

Therefore we have the relation

$$y_t = \alpha(1 - \theta) + \beta(1 - \theta)x_t + \theta y_{t-1} + v_t$$

where the disturbance term is given by

$$v_t = (u_t - \theta u_{t-1})$$

with $E[v_t] = E[(u_t - \theta u_{t-1})] = E[u_t] - \theta E[u_{t-1}] = 0$. ***The thing to notice is that even if the original disturbances u_t are not autocorrelated (in particular, $E[u_t u_{t-1}] = 0$), the new disturbances v_t are composed of two successive u-values, and hence <u>must</u> exhibit autocorrelation.*** This can easily be demonstrated as follows:

$$E[v_t v_{t-1}]=E[(u_t - \theta u_{t-1})(u_{t-1} - \theta u_{t-2})]=E[u_t u_{t-1}-\theta u_{t-1}^2-\theta u_t u_{t-2}+\theta^2 u_{t-1}u_{t-2}] = E[- \theta u_{t-1}^2] = -\theta E[u_{t-1}^2] = -\theta\sigma_u^2$$

So successive values of the v-series are negatively correlated: $E[v_t v_{t-1}] = -\theta\sigma_u^2 < 0$.

This result provides a contrast to the partial adjustment model: if the random disturbances in the behavioural equations of each model are free of autocorrelation, so is the disturbance term in the final equation of the partial adjustment model, but not of the adaptive expectations model. As stated earlier, the error term in the final equation of the partial adjustment model is simply (1-γ) times the disturbance term introduced into the equation for y_t^*, whereas the error term in the final equation of the adaptive expectations model is a moving average of the original disturbance, ***as a result of the transformation used to eliminate the unobservable variable,*** so the disturbance term in the adaptive expectations model is generally autocorrelated.

Although the systematic parts of the final equations are identical in the partial adjustment and adaptive expectations models, the correlation pattern of the disturbances is not, and this can be exploited in empirical work.

13.7. <u>The rational expectations hypothesis</u>

In Section 13.5, we considered one basic approach to the modelling of unobservable expectations variables which was based on the assumption that expectations are formed by extrapolating from past experience. ***Thus, forecasts of the future value of a variable were calculated from its current and past values.***

This approach remains the appropriate one if the variable about which expectations are formed is an <u>exogenous</u> variable, since by definition, our theories and models do not describe the determination of the values of exogenous variables. ***However, if the variable about which expectations are formed is an <u>endogenous</u> variable, then it is more appropriate to assume that expectations are formed as if agents anticipate the workings of the model that determines the value of the variable.***

For example, a change in the level of excise tax on a commodity will have an effect on the price of the commodity, and if either suppliers or consumers wish to predict this effect they will do better to consider relevant demand and supply factors than to base a prediction on past prices. ***The discussion in this section proceeds in the context of a simple example, namely that of the market for an agricultural commodity.***

We assume that the farmers who produce this commodity must make their planting decisions *before* they know the price that they will receive for the crop at harvest time. ***If a high price for the crop is anticipated, then a substantial harvest will be planned, but if a low price is expected, then planting will be reduced.***

Using a linear approximation, we can represent the farmers' supply function as

$$Q_t = \beta_0 + \beta_1 \overline{P}_t \qquad \beta_1 > 0$$

where Q_t is the quantity supplied to the market, $\overline{P}_t$ is the expectation of the market clearing price P_t formed at the time of planting, and we are ignoring random disturbances and other exogenous influences on supply (such as climactic factors) for the moment.

These ignored factors are among those which contribute to uncertainty about price, but <u>demand</u> for the product may also be uncertain. In the face of this uncertainty, we might try to predict the price by looking at past prices, in which case the framework of Section 13.5 is relevant: for example, the 'naive' prediction rule $\overline{P}_t = P_{t-1}$ might be used, in which case we end up with the supply function of a simple 'cobweb' model.

However, the rational expectations hypothesis postulates that price expectations are essentially the same as the predictions of the relevant demand and supply model. Let us complete the model with the simple demand function

$$Q_t = \alpha_0 + \alpha_1 P_t \qquad \alpha_1 < 0, \alpha_0 > \beta_0$$

and assume that the price adjusts so as to clear the market. The hypothesis is that expectations are formed as if farmers correctly anticipate the operation of the market. An expression for the required value of $\overline{P}_t$ can easily be deduced as follows. Equating demand and supply gives

$$\alpha_0 + \alpha_1 P_t = \beta_0 + \beta_1 \overline{P}_t$$

Thus, the relation between actual and expected price is easily seen to be

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t \qquad (**)$$

The value of $\overline{P}_t$ such that this expectation is fulfilled, $\overline{P}_t = P_t$, is given by solving

$$\overline{P}_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t$$

We get

$$[(\alpha_1 - \beta_1)/\alpha_1]\overline{P}_t = (\beta_0 - \alpha_0)/\alpha_1$$

so

$$\overline{P}_t = (\beta_0 - \alpha_0)/(\alpha_1 - \beta_1)$$

or

$$\overline{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1)$$

If you substitute this value of $\overline{P}_t$ into equation (**) above, you will see after some algebra that you get $P_t = \overline{P}_t$, as required. This price expectation leads to the supply of the quantity

$$Q_t = \beta_0 + \beta_1 \overline{P}_t = \beta_0 + \beta_1[(\alpha_0 - \beta_0)/(\beta_1 - \alpha_1)] = [\beta_0(\beta_1 - \alpha_1) + \beta_1(\alpha_0 - \beta_0)]/(\beta_1 - \alpha_1)$$
$$= (\alpha_0\beta_1 - \alpha_1\beta_0)/(\beta_1 - \alpha_1)$$

i.e.

$$Q_t = (\alpha_0\beta_1 - \alpha_1\beta_0)/(\beta_1 - \alpha_1)$$

The buyers' demand function tells us that, when faced with this quantity, the market will clear at a price given by

$$(\alpha_0\beta_1 - \alpha_1\beta_0)/(\beta_1 - \alpha_1) = \alpha_0 + \alpha_1 P_t$$

This can be solved to give

$$P_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1)$$

which is exactly what the price was (rationally) expected to be. In other words, the rationally expected price is a 'self-fulfilling prophecy'.

This exact self-fulfilling nature of rational expectations is modified once random disturbances enter the model. The stochastic version of our model is

$$Q_t = \alpha_0 + \alpha_1 P_t + u_{1t}$$
$$Q_t = \beta_0 + \beta_1 \overline{P}_t + u_{2t}$$

and so the relation between actual and expected price becomes

$$\alpha_0 + \alpha_1 P_t + u_{1t} = \beta_0 + \beta_1 \overline{P}_t + u_{2t}$$

which rearranges to give

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t + (u_{2t} - u_{1t})/\alpha_1$$

Expectations formed at time t - 1 consistent with this relation satisfy

$$\overline{P}_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t + (\overline{u}_{2t} - \overline{u}_{1t})/\alpha_1$$

where $\bar{u}_{1t}$ and $\bar{u}_{2t}$ are the predicted values of the disturbance terms. ***If the disturbances are not autocorrelated, so that information up to time t-1 does not help in forecasting their values at time t, then the best forecasts of u_{1t} and u_{2t} are simply their mean values, namely zero.*** Then we end up with the same expression as before for $\bar{P}_t$:

$$\bar{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1)$$

Now, however, the actual price deviates from this expectation as a result of these unanticipated random shocks, since the preceding relations give

$$P_t = \bar{P}_t + (u_{2t} - u_{1t})/\alpha_1$$

(You are asked to prove this in exercise 1 for Lecture 13 - the solution is attached). For example, if weather conditions are unusually bad, and the harvest is reduced ($u_{2t} < 0$), then the market clearing price is higher than anticipated ($u_{2t}/\alpha_1 > 0$, remembering that $\alpha_1 < 0$). ***Thus, there will be errors in the price forecasts, but these will be purely random, and exhibit no systematic pattern.*** Having observed this higher-than-anticipated price, rational farmers' behaviour in the following period will not change.

We shall now show how the rational expectations hypothesis can give rise to dynamic terms in equations. We can do this by adding a single exogenous variable to the model. Thus, let us assume that the level of demand is affected by the variable X_t, which might represent consumers' income or the price of an imported substitute for the commodity, for example. Now the model is

$$Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t}$$
$$Q_t = \beta_0 + \beta_1 \bar{P}_t + u_{2t}$$

Equating right hand sides gives

$$\alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t} = \beta_0 + \beta_1 \bar{P}_t + u_{2t}$$

which upon rearranging gives

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\bar{P}_t - (\alpha_2/\alpha_1)X_t + (u_{2t} - u_{1t})/\alpha_1$$

This would be the 'reduced form' equation for P_t ***if*** $\bar{P}_t$ ***were an observable exogenous variable, but under the rational expectations hypothesis we simply use it to tell us how the unobservable variable*** $\bar{P}_t$ ***is formed.*** Again assuming that the random disturbances are not autocorrelated, expectations consistent with the model satisfy

$$\bar{P}_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\bar{P}_t - (\alpha_2/\alpha_1)\bar{X}_t$$

where $\bar{X}_t$ is the expectation of X_t formed at time t-1. This can easily be solved for $\bar{P}_t$, giving

$$\bar{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1) + [\alpha_2/(\beta_1 - \alpha_1)]\bar{X}_t$$

This result can be interpreted as follows: ***knowing that demand, and hence the market price, is influenced by the variable X, it is rational for farmers to take this into account when forming price expectations, by attempting to predict X_t.***

Errors in forecasting the exogenous variable now contribute an additional component to the error in the rational expectation, because we have

$$P_t = \bar{P}_t - (\alpha_2/\alpha_1)(X_t - \bar{X}_t) + (u_{2t} - u_{1t})/\alpha_1$$

(You are asked to prove this equation in exercise 2 for Lecture 13). ***However, the error*** $P_t - \bar{P}_t$ ***will again be purely random if past information on X is used in an optimal way, so that its forecast error*** $X_t - \bar{X}_t$ ***is purely random.*** For example, if X_t follows a first-order autoregression

$$X_t = \rho X_{t-1} + \varepsilon_t$$

then as we saw earlier, the best forecast calclated at time t-1 is

$$\overline{X}_t = \rho X_{t-1}$$

On substituting for $\overline{X}_t$, we now obtain an expression for $\overline{P}_t$ in terms of X_{t-1} and various parameters:

$$\overline{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1) + [\alpha_2/(\beta_1 - \alpha_1)]\overline{X}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1) + [\alpha_2\rho/(\beta_1 - \alpha_1)]X_{t-1}$$

On substituting this in turn into the 'reduced form' equation

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t - (\alpha_2/\alpha_1)X_t + (u_{2t} - u_{1t})/\alpha_1$$

which we derived above, and rearranging, we get a dynamic equation linking P and X:

$$P_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1) - (\alpha_2/\alpha_1)X_t + [\beta_1\alpha_2\rho/\alpha_1(\beta_1 - \alpha_1)]X_{t-1} + (u_{2t} - u_{1t})/\alpha_1$$

There is no obvious dynamic element in the original model (eg. lags in adjustment). But there is a time delay between the formation of price <u>expectations</u>, and the realisation of <u>actual</u> price, and this is enough to give another potential source of dynamic models.

13.8. Some practice problems

Here are some problems to help you become familiar with the algebraic issues surrounding dynamic models. Full solutions to all the problems are attached.

Exercise 1: On page 9 of the handout for Lecture 13 Part 2, it was asserted that

$$P_t = \overline{P}_t + (u_{2t} - u_{1t})/\alpha_1$$

when the stochastic model is of the form

$$Q_t = \alpha_0 + \alpha_1 P_t + u_{1t}$$
$$Q_t = \beta_0 + \beta_1 \overline{P}_t + u_{2t}$$

Prove that this is true by substituting $\overline{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1)$ into the equation

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t + (u_{2t} - u_{1t})/\alpha_1$$

(see page 8) and rearranging.

Exercise 2: On page 9 of the handout for Lecture 13 Part 2, it was asserted that

$$P_t = \overline{P}_t - (\alpha_2/\alpha_1)(X_t - \overline{X}_t) + (u_{2t} - u_{1t})/\alpha_1$$

when the stochastic model is of the form

$$Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t}$$
$$Q_t = \beta_0 + \beta_1 \overline{P}_t + u_{2t}$$

Prove that this is true by substituting $\overline{P}_t = (\alpha_0 - \beta_0)/(\beta_1 - \alpha_1) + [\alpha_2/(\beta_1 - \alpha_1)]\overline{X}_t$ into the equation

$$P_t = (\beta_0 - \alpha_0)/\alpha_1 + (\beta_1/\alpha_1)\overline{P}_t - (\alpha_2/\alpha_1)X_t + (u_{2t} - u_{1t})/\alpha_1$$

(see page 9) and rearranging.

Exercise 3: Consider the market for a commodity produced in an annual crop. The quantity demanded depends on the current price:

$$Q_t = \alpha_0 + \alpha_1 P_t$$

The quantity supplied this year is a function of last year's price:

$$Q_t = \beta_0 + \beta_1 P_{t-1}$$

and the market is cleared in every year. Under what conditions on the parameters is the system stable? Now suppose that the quantity supplied is a function of this year's expected price, where price expectations are formed according to the adapative expectations hypothesis. Show that the final equation for P_t is

$$P_t = \frac{(1-\theta)(\beta_0 - \alpha_0)}{\alpha_1} + \left\{\frac{\beta_1}{\alpha_1}(1-\theta) + \theta\right\}P_{t-1}$$

Exercise 4: Obtain the reduced form and final equation for Y_t, and the stability condition for the following model:

$$C_t = \alpha + \beta Y_t + \gamma C_{t-1}$$
$$Y_t = C_t + I_t$$

endogenous variables: C_t, Y_t

exogenous variables: I_t

Adding a disturbance term u_t to the consumption function, derive the final equation for C_t in the stochastic version of the above model. Check that the final equation for C_t has the same autoregressive structure as that for Y_t.

Exercise 5: Obtain the final equation for Y_t and the stability condition for the following model:

$$C_t = \alpha + \beta Y_t$$
$$Y_t = C_t + I_t$$
$$I_t = \gamma \Delta Y_t + G_t$$

endogenous variables: C_t, Y_t, I_t

exogenous variables: G_t

Adding a disturbance term u_t to the consumption function, derive the final equation for C_t in the stochastic version of the above model. Check that the final equation for C_t has the same autoregressive structure as that for Y_t.

(End of Lecture 13)

Solutions to practice problems for Lecture 13

Exercise 1:

Substituting $\bar{P}_t = \dfrac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1}$ into the

equation $$P_t = \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1}{\alpha_1}\bar{P}_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

gives

$$P_t = \frac{\beta - \alpha_0}{\alpha_1} + \frac{\beta_1}{\alpha_1}\left(\frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1}\right) + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \frac{(\beta_0 - \alpha_0)(\beta_1 - \alpha_1) + \beta_1(\alpha_0 - \beta_0)}{\alpha_1(\beta_1 - \alpha_1)} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \frac{\cancel{\beta_0\beta_1} - \alpha_1\beta_0 - \cancel{\alpha_0\beta_1} + \alpha_0\alpha_1 + \cancel{\beta_1\alpha_0} - \cancel{\beta_1\beta_0}}{\alpha_1(\beta_1 - \alpha_1)} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \frac{\cancel{\alpha_1}(\alpha_0 - \beta_0)}{\cancel{\alpha_1}(\beta_1 - \alpha_1)} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

QED.

Exercise 2:

Substituting $\bar{P}_t = \dfrac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \dfrac{\alpha_2}{\beta_1 - \alpha_1} \bar{X}_t$

into the equation

$$P_t = \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1}{\alpha_1} \bar{P}_t - \frac{\alpha_2}{\alpha_1} X_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

gives

$$P_t = \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1}{\alpha_1} \left(\frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \frac{\alpha_2}{\beta_1 - \alpha_1} \bar{X}_t \right) - \frac{\alpha_2}{\alpha_1} X_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} + \frac{\beta_1 \alpha_2 \bar{X}_t}{\alpha_1 (\beta_1 - \alpha_1)} - \frac{\alpha_2}{\alpha_1} X_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t + \frac{\beta_1 \alpha_2 \bar{X}_t - (\beta_1 - \alpha_1) \alpha_2 X_t}{\alpha_1 (\beta_1 - \alpha_1)} - \frac{\alpha_2}{(\beta_1 - \alpha_1)} \bar{X}_t + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t + \frac{\beta_1 \alpha_2 \bar{X}_t - (\beta_1 - \alpha_1) \alpha_2 X_t - \alpha_2 \alpha_1 \bar{X}_t}{\alpha_1 (\beta_1 - \alpha_1)} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t + \frac{\cancel{(\beta_1 - \alpha_1)} \alpha_2 (\bar{X}_t - X_t)}{\alpha_1 \cancel{(\beta_1 - \alpha_1)}} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t + \frac{\alpha_2 (\bar{X}_t - X_t)}{\alpha_1} + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

$$= \bar{P}_t - \frac{\alpha_2}{\alpha_1} (X_t - \bar{X}_t) + \frac{u_{2t} - u_{1t}}{\alpha_1}$$

QED.

Exercise 3:

Setting demand equal to supply, and rearranging, we get the final equation for price:

$$\alpha_0 + \alpha_1 P_t = \beta_0 + \beta_1 P_{t-1}$$

$$\Rightarrow \boxed{P_t = \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1}{\alpha_1} P_{t-1}}$$

The stability condition is therefore

$$\left| \frac{\beta_1}{\alpha_1} \right| < 1$$

or

$$\boxed{-1 < \frac{\beta_1}{\alpha_1} < 1}$$

If quantity supplied is a function of <u>this</u> years' expected price, we have

$$Q_t = \beta_0 + \beta_1 \overline{P}_t$$

If price expectations are formed according to the adaptive expectations hypothesis, we must have

$$\overline{P}_t = (1-\theta) P_{t-1} + \theta \overline{P}_{t-1}$$

Substituting into the supply function gives

$$Q_t = \beta_0 + \beta_1 (1-\theta) P_{t-1} + \beta_1 \theta \overline{P}_{t-1}$$

Now when we equate demand and supply we get

$$\alpha_0 + \alpha_1 P_t = \beta_0 + \beta_1 (1-\theta) P_{t-1} + \beta_1 \theta \overline{P}_{t-1}$$

so the new equation for P_t is

$$\boxed{P_t = \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1 (1-\theta)}{\alpha_1} P_{t-1} + \frac{\beta_1 \theta}{\alpha_1} \overline{P}_{t-1}}$$

To get the new final equation for P_t, we need to get rid of $\overline{P}_{t-1}$. We do it by the usual "trick" for getting rid of unobservable variables in adaptive expectations models:

$$P_t - \theta P_{t-1} = \left\{ \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1(1-\theta)}{\alpha_1} P_{t-1} + \frac{\beta_1 \theta}{\alpha_1} \overline{P}_{t-1} \right\}$$

$$- \theta \left\{ \frac{\beta_0 - \alpha_0}{\alpha_1} + \frac{\beta_1(1-\theta)}{\alpha_1} P_{t-2} + \frac{\beta_1 \theta}{\alpha_1} \overline{P}_{t-2} \right\}$$

$$= \frac{(1-\theta)(\beta_0 - \alpha_0)}{\alpha_1} + \frac{\beta_1}{\alpha_1}(1-\theta) P_{t-1} - \frac{\beta_1}{\alpha_1}\theta(1-\theta)P_{t-2}$$

$$+ \frac{\beta_1}{\alpha_1}\theta\left(\overline{P}_{t-1} - \theta \overline{P}_{t-2}\right)$$

$$= \frac{(1-\theta)(\beta_0 - \alpha_0)}{\alpha_1} + \frac{\beta_1}{\alpha_1}(1-\theta) P_{t-1}$$

So we get the following final equation for P_t:

$$\boxed{P_t = \frac{(1-\theta)(\beta_0 - \alpha_0)}{\alpha_1} + \left\{ \frac{\beta_1}{\alpha_1}(1-\theta) + \theta \right\} P_{t-1}}$$

Exercise 4:

The reduced form equation for Y_t is obtained by substituting the equation for C_t into the equation for Y_t:

$$Y_t = \alpha + \beta Y_t + \gamma C_{t-1} + I_t$$

$$\Rightarrow \boxed{Y_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta} C_{t-1} + \frac{I_t}{1-\beta}}$$

From the income identity,

$$Y_{t-1} = C_{t-1} + I_{t-1}$$

so $$C_{t-1} = Y_{t-1} - I_{t-1}$$

Substituting this into the reduced form gives the required final equation:

$$\boxed{Y_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta} Y_{t-1} - \frac{\gamma}{1-\beta} I_{t-1} + \frac{I_t}{1-\beta}}$$

The stability condition is $-1 < \frac{\gamma}{1-\beta} < 1$.

After adding a disturbance term to the consumption function, we get the final equation for C_t by substituting $Y_t = C_t + I_t$ into it:

$$C_t = \alpha + \beta C_t + \beta I_t + \gamma C_{t-1} + u_t$$

Rearranging:

$$\boxed{C_t = \frac{\alpha}{1-\beta} + \frac{\beta}{1-\beta} I_t + \frac{\gamma}{1-\beta} C_{t-1} + \frac{u_t}{1-\beta}}$$

This has the same autoregressive structure as the final equation for Y_t.

Exercise 5:

We have $I_t = \gamma \Delta Y_t + G_t = \gamma Y_t - \gamma Y_{t-1} + G_t$

Substituting this and the equation for C_t into the equation for Y_t gives

$$Y_t = \alpha + \beta Y_t + \gamma Y_t - \gamma Y_{t-1} + G_t$$

so the final equation is

$$\boxed{Y_t = \frac{\alpha}{(1-\beta-\gamma)} - \frac{\gamma}{(1-\beta-\gamma)} Y_{t-1} + \frac{G_t}{(1-\beta-\gamma)}}$$

The stability condition is

$$\boxed{-1 < \frac{\gamma}{(1-\beta-\gamma)} < 1}$$

Adding a disturbance term u_t to the consumption function gives us the following model:

$$C_t = \alpha + \beta Y_t + u_t$$

$$Y_t = C_t + I_t$$

$$I_t = \gamma \Delta Y_t + G_t$$

We find the final equation for C_t using the "trick" described on page 8 of Lecture 13 Part 1. We have

$$C_t + \frac{\gamma}{1-\beta-\gamma} C_{t-1} = (\alpha + \beta Y_t + u_t) + \frac{\gamma}{1-\beta-\gamma}(\alpha + \beta Y_{t-1} + u_{t-1})$$

$$= \frac{(1-\beta-\gamma)\alpha + \alpha\gamma}{1-\beta-\gamma} + \beta \underbrace{\left(Y_t + \frac{\gamma}{1-\beta-\gamma} Y_{t-1}\right)}_{\text{use final equation for } Y_t \text{ to get rid of this}} + u_t + \frac{\gamma}{1-\beta-\gamma} u_{t-1}$$

$$= \frac{(1-\beta)\alpha}{1-\beta-\gamma} + \beta\left(\frac{\alpha + G_t}{1-\beta-\gamma}\right) + u_t + \frac{\gamma}{1-\beta-\gamma} u_{t-1}$$

$$= \frac{\alpha + G_t}{1-\beta-\gamma} + u_t + \frac{\gamma}{1-\beta-\gamma} u_{t-1}$$

So the final equation for C_t is

$$\boxed{C_t = \frac{\alpha + G_t}{1-\beta-\gamma} - \frac{\gamma}{1-\beta-\gamma} C_{t-1} + u_t + \frac{\gamma}{1-\beta-\gamma} u_{t-1}}$$

This has same autoregressive structure as final equation for Y_t.